This chapter is devoted to annual cash flow analysis—the second of the three major analysis techniques. As we've said, alternatives must be resolved into such a form that they may be compared. This means we must use the equivalence concept to convert from a cash flow representing the alternative into some equivalent sum or equivalent cash flow.

With present worth analysis, we resolved an alternative into an equivalent cash sum. This might have been an equivalent present worth of cost, an equivalent present worth of benefit, or an equivalent net present worth. But instead of computing equivalent present sums, we could compare alternatives based on their equivalent annual cash flows. Depending on the particular situation, we may wish to compute the equivalent uniform annual cost (EUAC), the equivalent uniform annual benefit (EUAB), or their difference (EUAB—EUAC).

To prepare for a discussion of annual cash flow analysis, we will review some annual cash flow calculations, then examine annual cash flow criteria. Following this, we will proceed with annual cash flow analysis.

**Annual Cash Flow Calculations**

**Resolving a Present Cost to an Annual Cost**

Equivalence techniques were used in prior chapters to convert money, at one point in time, to some equivalent sum or series. In annual cash flow analysis, the goal is to convert money to
an equivalent uniform annual cost or benefit. The simplest case is to convert a present sum \( p \) to a series of equivalent uniform end-of-period cash flows. This is illustrated in Example 6-1.

**EXAMPLE 6-1**

A woman bought $1000 worth of furniture for her home. If she expects it to last ten years, what will be her equivalent uniform annual cost if interest is 7%?

\[
\begin{align*}
\text{Solution:} \\
\text{Equivalent uniform annual cost} &= P(A/P, i, n) \\
&= 1000(A/P, 7\%, 10) \\
&= $142.40
\end{align*}
\]

Her equivalent uniform annual cost is $142.40.

**Treatment of Salvage Value**

In a situation where there is a salvage value, or future value at the end of the useful life of an asset, the result is to decrease the equivalent uniform annual cost.

**EXAMPLE 6-2**

The woman in Ex. 6-1 now believes she can resell the furniture at the end of ten years for $200. Under these circumstances, what is her equivalent uniform annual cost?

\[
\begin{align*}
\text{Resale value} S &= 200 \\
\text{Solution:} &\text{ For this situation, the problem may be solved by each of three different calculations, as follows:}
\end{align*}
\]
Ex. 6-2, Solution One:

\[
EUAC = P(A/P, i, n) - S(A/F, i, n)
\]

\[
= 1000(A/P, 7\%, 10) - 200(A/F, 7\%, 10)
\]

\[
= 1000(0.1424) - 200(0.0724)
\]

\[
= 142.40 - 14.48 = $127.92
\]

This method reflects the Annual cost of the cash disbursement minus the Annual benefit of the future resale value.

Ex. 6-2, Solution Two: Equation 6-1 describes a relationship that may be modified by an identity presented in Chapter 4:

\[
(A/P, i, n) = (A/F, i, n) + i
\]

(6-2)

Substituting this into Eq. 6-1 gives:

\[
EUAC = P(A/F, i, n) + Pi - S(A/F, i, n)
\]

\[
= (P - S)(A/F, i, n) + Pi
\]

\[
= (1000 - 200)(A/F, 7\%, 10) + 1000(0.07)
\]

\[
= 800(0.0724) + 70 = 57.92 + 70
\]

\[
= $127.92
\]

This method computes the equivalent annual cost due to the unrecovered $800 when the furniture is sold, and adds annual interest on the $1000 investment.

Ex. 6-2, Solution Three: If the value for \((A/F, i, n)\) from Eq. 6-2 is substituted into Eq. 6-1, we obtain:

\[
EUAC = P(A/P, i, n) - S(A/P, i, n) + Si
\]

\[
= (P - S)(A/P, i, n) + Si
\]

\[
= (1000 - 200)(A/P, 7\%, 10) + 200(0.07)
\]

\[
= 800(0.1424) + 14 = 113.92 + 14 = $127.92
\]

This method computes the annual cost of the $800 decline in value during the ten years, plus interest on the $200 tied up in the furniture as the salvage value.
Example 6-2 illustrates that when there is an initial disbursement $P$ followed by a salvage value $S$, the annual cost may be computed in three different ways:

1. \[ \text{EUAC} = P(A/P, i, n) - S(A/F, i, n) \quad (6-1) \]
2. \[ \text{EUAC} = (P - S)(A/F, i, n) + Pi \quad (6-3) \]
3. \[ \text{EUAC} = (P - S)(A/P, i, n) + Si \quad (6-4) \]

Each of the three calculations gives the same results. In practice, the first and third methods are most commonly used.

EXAMPLE 6-3

Bill owned a car for five years. One day he wondered what his uniform annual cost for maintenance and repairs had been. He assembled the following data:

<table>
<thead>
<tr>
<th>Year</th>
<th>Maintenance and repair cost for year</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$45</td>
</tr>
<tr>
<td>2</td>
<td>90</td>
</tr>
<tr>
<td>3</td>
<td>180</td>
</tr>
<tr>
<td>4</td>
<td>135</td>
</tr>
<tr>
<td>5</td>
<td>225</td>
</tr>
</tbody>
</table>

Compute the equivalent uniform annual cost (EUAC) assuming 7% interest and end-of-year disbursements.

**Solution:** The EUAC may be computed for this irregular series of payments in two steps:

1. Compute the present worth of cost for the five years using single payment present worth factors.

   PW of cost = $45(P/F, 7\%, 1) + 90(P/F, 7\%, 2) + 180(P/F, 7\%, 3) 
               \quad + 135(P/F, 7\%, 4) + 225(P/F, 7\%, 5) 
               \quad = 45(0.9346) + 90(0.8734) + 180(0.8163) 
               \quad + 135(0.7629) + 225(0.7130) 
               \quad = $531$

2. With the PW of cost known, compute EUAC using the capital recovery factor.

   \[ \text{EUAC} = 531(A/P, 7\%, 5) = 531(0.2439) = $130 \]
EXAMPLE 6-4

Bill reexamined his calculations and found that he had reversed the Year 3 and 4 maintenance and repair costs in his table. The correct table is:

<table>
<thead>
<tr>
<th>Year</th>
<th>Maintenance and repair cost for year</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$ 45</td>
</tr>
<tr>
<td>2</td>
<td>90</td>
</tr>
<tr>
<td>3</td>
<td>135</td>
</tr>
<tr>
<td>4</td>
<td>180</td>
</tr>
<tr>
<td>5</td>
<td>225</td>
</tr>
</tbody>
</table>

Recompute the EUAC.

Solution: This time the schedule of disbursements is an arithmetic gradient series plus a uniform annual cost, as follows:

\[ EUAC = 45 + 45(A/G, 7\%, 5) \]
\[ = 45 + 45(1.865) \]
\[ = $129 \]

Since the timing of the Ex. 6-3 and 6-4 expenditures is different, we would not expect to obtain the same EUAC.

The examples have shown four essential points concerning cash flow calculations:

1. **There is a direct relationship between the present worth of cost and the equivalent uniform annual cost.** It is

   \[ EUAC = (PW \text{ of cost})(A/P, i, n) \]

2. **In a problem, an expenditure of money increases the EUAC, while a receipt of money (like selling something for its salvage value) decreases EUAC.**
3. When there are irregular cash disbursements over the analysis period, a convenient method of solution is to first determine the PW of cost; then, using the equation in Item 1 above, the EUAC may be calculated.

4. Where there is an increasing uniform gradient, EUAC may be rapidly computed using the arithmetic gradient uniform series factor, \((A/G,i,n)\).

Annual Cash Flow Analysis

The criteria for economic efficiency are presented in Table 6-1. One notices immediately that the table is quite similar to Table 5-1. In the case of fixed input, for example, the present worth criterion is \(\text{maximize PW of benefits}\), and the annual cost criterion is \(\text{maximize equivalent uniform annual benefits}\). It is apparent that, if you are maximizing the present worth of benefits, simultaneously you must be maximizing the equivalent uniform annual benefits. This is illustrated in Example 6-5.

<table>
<thead>
<tr>
<th>Situation</th>
<th>Criterion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed input</td>
<td>Amount of money or other input resources</td>
</tr>
<tr>
<td></td>
<td>is fixed</td>
</tr>
<tr>
<td></td>
<td>Maximize equivalent uniform benefits (maximize EUAB)</td>
</tr>
<tr>
<td>Fixed output</td>
<td>There is a fixed task, benefit, or other output</td>
</tr>
<tr>
<td></td>
<td>to be accomplished</td>
</tr>
<tr>
<td></td>
<td>Minimize equivalent uniform annual cost (minimize</td>
</tr>
<tr>
<td></td>
<td>EUAC)</td>
</tr>
<tr>
<td>Neither input</td>
<td>Neither amount of money, or other inputs, nor</td>
</tr>
<tr>
<td>nor output is</td>
<td>amount of benefits, nor other outputs, is fixed</td>
</tr>
<tr>
<td>fixed</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Maximize (EUAB – EUAC)</td>
</tr>
</tbody>
</table>
EXAMPLE 6-5

A firm is considering which of two devices to install to reduce costs in a particular situation. Both devices cost $1000 and have useful lives of five years with no salvage value. Device A can be expected to result in $300 savings annually. Device B will provide cost savings of $400 the first year but will decline $50 annually, making the second year savings $350, the third year savings $300, and so forth. With interest at 7%, which device should the firm purchase?

**Solution:**

Device A:

EUAB = $300

Device B:

\[
EUAB = 400 - 50(A/G,7\%,5) = 400 - 50(1.865)
\]

\[
= 306.75
\]

To maximize EUAB, select device B. ■

Example 6-5 was previously presented as Ex. 5-1 where we found:

PW of benefits A = \(300(P/A,7\%,5) = 300(4.100) = 1230\)

This is converted to EUAB by multiplying by the capital recovery factor:

\[
EUAB_A = 1230(A/P,7\%,5) = 1230(0.2439) = 300
\]

PW of benefits B = \(400(P/A,7\%,5) - 50(P/G,7\%,5)\)

\[
= 400(4.100) - 50(7.647) = 1257.65
\]

and, hence,

\[
EUAB_B = 1257.65(A/P,7\%,5) = 1257.65(0.2439)
\]

\[
= 306.75
\]

We see, therefore, that it is easy to convert the present worth analysis results into the annual cash flow analysis results. We could go from annual cash flow to present worth just as easily using the series present worth factor. And, of course, both methods indicate the same device B as the preferred alternative.
EXAMPLE 6-6

Three alternatives are being considered for improving an operation on the assembly line along with the do-nothing alternative. The cost of the equipment varies as do their annual benefits compared to the present situation. Each of Plans A, B, and C has a ten-year life and a scrap value equal to 10% of its original cost.

<table>
<thead>
<tr>
<th></th>
<th>Plan A</th>
<th>Plan B</th>
<th>Plan C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Installed cost of equipment</td>
<td>$15,000</td>
<td>$25,000</td>
<td>$33,000</td>
</tr>
<tr>
<td>Material and labor savings per year</td>
<td>14,000</td>
<td>9,000</td>
<td>14,000</td>
</tr>
<tr>
<td>Annual operating expenses</td>
<td>8,000</td>
<td>6,000</td>
<td>6,000</td>
</tr>
<tr>
<td>End-of-useful life scrap value</td>
<td>1,500</td>
<td>2,500</td>
<td>3,300</td>
</tr>
</tbody>
</table>

If interest is 8%, which plan, if any, should be adopted?

Solution: Since neither installed cost nor output benefits are fixed, the economic criterion is to maximize (EUAB – EUAC).

<table>
<thead>
<tr>
<th></th>
<th>Plan A</th>
<th>Plan B</th>
<th>Plan C</th>
<th>Do nothing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equivalent uniform annual benefit (EUAB):</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Material and labor per year</td>
<td>$14,000</td>
<td>$9,000</td>
<td>$14,000</td>
<td>$0</td>
</tr>
<tr>
<td>Scrap value (A/F,8%,10)</td>
<td>104</td>
<td>172</td>
<td>228</td>
<td>0</td>
</tr>
<tr>
<td>EUAB =</td>
<td>$14,104</td>
<td>$9,172</td>
<td>$14,228</td>
<td>$0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Plan A</th>
<th>Plan B</th>
<th>Plan C</th>
<th>Do nothing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equivalent uniform annual cost (EUAC):</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Installed cost (A/P,8%,10)</td>
<td>$2,235</td>
<td>$3,725</td>
<td>$4,917</td>
<td>$0</td>
</tr>
<tr>
<td>Annual operating expenses</td>
<td>8,000</td>
<td>6,000</td>
<td>6,000</td>
<td>0</td>
</tr>
<tr>
<td>EUAC =</td>
<td>$10,235</td>
<td>$9,725</td>
<td>$10,917</td>
<td>$0</td>
</tr>
<tr>
<td>(EUAB – EUAC) =</td>
<td>$3,869</td>
<td>-$553</td>
<td>$3,311</td>
<td>$0</td>
</tr>
</tbody>
</table>

Based on our criterion of maximizing (EUAB – EUAC), Plan A is the best of the four alternatives. We note, however, that since the do-nothing alternative has (EUAB – EUAC) = 0, it is a more desirable alternative than Plan B. ■
Analysis Period

In the last chapter, we saw that the analysis period was an important consideration in computing present worth comparisons. It was essential that a common analysis period be used for each alternative. In annual cash flow comparisons, we again have the analysis period question. Example 6-7 will help in examining the problem.

**EXAMPLE 6-7**

Two pumps are being considered for purchase. If interest is 7%, which pump should be bought?

<table>
<thead>
<tr>
<th></th>
<th>Pump A</th>
<th>Pump B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial cost</td>
<td>$7000</td>
<td>$5000</td>
</tr>
<tr>
<td>End-of-useful-life salvage value</td>
<td>1500</td>
<td>1000</td>
</tr>
<tr>
<td>Useful life, in years</td>
<td>12</td>
<td>6</td>
</tr>
</tbody>
</table>

**Solution:** The annual cost for twelve years of Pump A can be found using Eq. 6-4:

\[
EUAC = (P - S)(A/P, i,n) + Si
\]

\[
= (7000 - 1500)(A/P,7%,12) + 1500(0.07)
\]

\[
= 5500(0.1259) + 105 = $797
\]

Now compute the annual cost for six years of Pump B:

\[
EUAC = (5000 - 1000)(A/P,7%,6) + 1000(0.07)
\]

\[
= 4000(0.2098) + 70 = $909
\]

For a common analysis period of twelve years, we need to replace Pump B at the end of its six-year useful life. If we assume that another Pump B' can be obtained that has the same $5000 initial cost, $1000 salvage value and six-year life, the cash flow will be as follows:
For the twelve-year analysis period, the annual cost for Pump B:

\[
EUAC = [5000 - 1000(P/F,7\%,6) + 5000(P/F,7\%,6) - 1000(P/F,7\%,12)] \times (A/P,7\%,12)
\]

\[
= [5000 - 1000(0.6663) + 5000(0.6663) - 1000(0.4440)] \times (0.1259)
\]

\[
= (5000 - 666 + 3331 - 444)(0.1259)
\]

\[
= (7211)(0.1259) = $909
\]

The annual cost of B for the six-year analysis period is the same as the annual cost for the twelve-year analysis period. This is not a surprising conclusion when one recognizes that the annual cost of the first six-year period is repeated in the second six-year period. Thus the lengthy calculation of EUAC for twelve years of Pump was not needed. By assuming that the shorter-life equipment is replaced by equipment with identical economic consequences, we have avoided a lot of calculations and the analysis period problem. Select Pump A. ■

Analysis Period Equal to Alternative Lives

When the analysis period for an economy study coincides with the useful life for each alternative, we have an ideal situation which causes no difficulties. The economy study is based on this analysis period.
Analysis Period a Common Multiple of Alternative Lives

When the analysis period is a common multiple of the alternative lives (for example, in Ex. 6-7, the analysis period was twelve years with six- and twelve-year alternative lives), a "replacement with an identical item with the same costs, performance, and so forth" is frequently assumed. This means that when an alternative has reached the end of its useful life, it is assumed to be replaced with an identical item. As shown in Ex. 6-7, the result is that the EUAC for Pump B with a six-year useful life is equal to the EUAC for the entire analysis period based on Pump B plus Replacement Pump B'.

Under these circumstances of identical replacement, it is appropriate to compare the annual cash flows computed for alternatives based on their own service lives. In Ex. 6-7, the annual cost for Pump A, based on its 12-year service life, was compared with the annual cost for Pump B, based on its six-year service life.

Analysis Period For a Continuing Requirement

Many times the economic analysis is to determine how to provide for a more or less continuing requirement. One might need to pump water from a well as a continuing requirement. There is no distinct analysis period. In this situation, the analysis period is assumed to be long but undefined.

If, for example, we had a continuing requirement to pump water and alternative Pumps A and B had useful lives of seven and eleven years, respectively, what should we do? The customary assumption is that Pump A's annual cash flow (based on a seven-year life) may be compared to Pump B's annual cash flow (based on an eleven-year life). This is done without much concern that the least common multiple of the seven- and eleven-year lives is 77 years. This comparison of "different-life" alternatives assumes identical replacement (with identical costs, performance, and so forth) when an alternative reaches the end of its useful life. Example 6-8 illustrates the situation.

EXAMPLE 6-8

Pump B in Ex. 6-7 is now believed to have a nine-year useful life. Assuming the same initial cost and salvage value, compare it with Pump A using the same 7% interest rate.

Solution: If we assume that the need for A or B will exist for some continuing period, the comparison of annual costs for the unequal lives is an acceptable technique. For twelve years of Pump A:

\[
EUAC = (7000 - 1500)(A/P, 7\%, 12) + 1500(0.07) = 797
\]
For nine years of Pump B:

\[
\text{EUAC} = (5000 - 1000)(A/P, 7\%, 9) + 1000(0.07) = 684
\]

For minimum EUAC, select Pump B.

---

**Infinite Analysis Period**

At times we have an alternative with a limited (finite) useful life in an infinite analysis period situation. The equivalent uniform annual cost may be computed for the limited life. The assumption of identical replacement (replacements have identical costs, performance, and so forth) is often appropriate. Based on this assumption, the same EUAC occurs for each replacement of the limited-life alternative. The EUAC for the infinite analysis period is therefore equal to the EUAC computed for the limited life. With identical replacement,

\[
\text{EUAC for infinite analysis period} = \text{EUAC for limited life}
\]

A somewhat different situation occurs when there is an alternative with an infinite life in a problem with an infinite analysis period:

\[
\text{EUAC for infinite analysis period} = P(A/P, i, \infty) + \text{any other annual costs}
\]

When \( n = \infty \), we have \( A = P_i \) and, hence, \( (A/P, i, \infty) \) equals \( i \).

\[
\text{EUAC for infinite analysis period} = P_i + \text{any other annual costs}
\]

---

**EXAMPLE 6-9**

In the construction of the aqueduct to expand the water supply of a city, there are two alternatives for a particular portion of the aqueduct. Either a tunnel can be constructed through a mountain, or a pipeline can be laid to go around the mountain. If there is a permanent need for the aqueduct, should the tunnel or the pipeline be selected for this particular portion of the aqueduct? Assume a 6% interest rate.

**Solution:**

<table>
<thead>
<tr>
<th></th>
<th>Tunnel through mountain</th>
<th>Pipeline around mountain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial cost</td>
<td>$5.5 million</td>
<td>$5 million</td>
</tr>
<tr>
<td>Maintenance</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Useful life</td>
<td>Permanent</td>
<td>50 years</td>
</tr>
<tr>
<td>Salvage value</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
**Tunnel:** For the tunnel, with its permanent life, we want \( (A/P, 6\%, \infty) \). For an infinite life, the capital recovery is simply interest on the invested capital. So \( (A/P, 6\%, \infty) = i \),

\[
EUAC = Pi = \$5.5 \text{ million}(0.06) \\
= \$330,000
\]

**Pipeline:**

\[
EUAC = \$5 \text{ million}(A/P, 6\%, 50) \\
= \$5 \text{ million}(0.0634) = \$317,000
\]

For fixed output, minimize EUAC. Select the pipeline.

The difference in annual cost between a long life and an infinite life is small unless an unusually low interest rate is used. In Example 6-9 the tunnel is assumed to be permanent. For comparison, compute the annual cost if an 85-year life is assumed for the tunnel?

\[
EUAC = \$5.5 \text{ million}(A/P, 6\%, 85) \\
= \$5.5 \text{ million}(0.0604) = \$332,000
\]

The difference in time between 85 years and infinity is great indeed, yet the difference in annual costs in Example 6-9 is very small.

**Some Other Analysis Period**

The analysis period in a particular problem may be something other than one of the four we have so far described. It may be equal to the life of either the shorter-life alternative, the longer-life alternative, or something entirely different. One must carefully examine the consequences of each alternative throughout the analysis period and, in addition, see what differences there might be in salvage values, and so forth, at the end of the analysis period.

**EXAMPLE 6-10**

Consider a situation where Alternative 1 has a 7-year life and a salvage value at the end of that time. The replacement cost at the end of 7 years may be more or less than the original cost. If the replacement is retired prior to 7 years, it will have a terminal value that exceeds the end-of-life salvage value. Alternative 2 has a 13-year life and a terminal value whenever it is retired. If the situation indicates that 10 years is the proper analysis period, set up the
equations to properly compute the EUAC for each alternative.

Solution:

Alternative 1:

\[
\text{EUAC}_1 = [ \text{Initial cost} + (\text{Replacement cost} - \text{Salvage value})(P/F,i,7) \\
- (\text{Terminal value})(P/F,i,10)] (A/P,i,10)
\]

Alternative 2:

\[
\text{EUAC}_2 = [ \text{Initial cost} - (\text{Terminal value})(P/F,i,10)] (A/P,i,10)
\]
Using Spreadsheets to Analyze Loans

Loan and bond payments are made by firms, agencies, and individual engineers. Usually, the payments in each period are constant. Spreadsheets make it easy to:

- Calculate the loan's amortization schedule
- Decide how a payment is split between principal and interest
- Find the balance due on a loan
- Calculate the number of payments remaining on a loan.

Building an Amortization Schedule

As illustrated in Chapter 4 and Appendix 1, an amortization schedule lists for each payment period: the loan payment, interest paid, principal paid, and remaining balance. For each period the interest paid equals the interest rate times the balance remaining from the period before. Then the principal payment equals the payment minus the interest paid. Finally, this principal payment is applied to the balance remaining from the previous period to calculate the new remaining balance. As a basis for comparison with spreadsheet loan functions, Figure 6-1 shows this calculation for Example 6-11.

EXAMPLE 6-11

An engineer wanted to celebrate graduating and getting a job by buying $2400 of new furniture. Luckily the store was offering 6-month financing at the low interest rate of 6% per year nominal (really ½% per month). Calculate the amortization schedule.

Solution

The first step is to calculate the monthly payment.

\[ A = 2400 \left( A/P,\frac{1}{2}\%,6 \right) = 2400 \cdot 0.1696 = 407.0 \]
<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2400</td>
<td>Initial balance</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.50%</td>
<td>i</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>N</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$407.03</td>
<td>payment</td>
<td>=-PMT(A2,A3,A1)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>month</td>
<td>interest</td>
<td>payment</td>
<td>balance</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>2400.00</td>
<td>=A1</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>12.00</td>
<td>395.03</td>
<td>2004.97</td>
</tr>
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<td>2</td>
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<td>397.00</td>
<td>1607.97</td>
</tr>
<tr>
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<td>3</td>
<td>8.04</td>
<td>398.99</td>
<td>1208.98</td>
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<td>4</td>
<td>6.04</td>
<td>400.98</td>
<td>807.99</td>
</tr>
<tr>
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<td>4.04</td>
<td>402.99</td>
<td>405.00</td>
</tr>
<tr>
<td>14</td>
<td>6</td>
<td>2.03</td>
<td>405.00</td>
<td>0.00</td>
</tr>
<tr>
<td>15</td>
<td></td>
<td>=$A$4-B14</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
<td>=$A$2*D13</td>
<td>payment - interest</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td></td>
<td></td>
<td>= rate*previous balance</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 6-1 Amortization schedule for furniture loan.

How Much to Interest? How Much to Principal?

For a loan with constant payments, we can answer these questions for any period without the full amortization schedule. For a loan with constant payments, the functions IPMT and PPMT directly answer these questions. For simple problems, both functions have four arguments (i,t,N,-P), where t is the time period being calculated. Both functions have optional arguments that permit adding a balloon payment (an F) and changing from end-of-period payments to beginning-of-period payments.

For example, consider period 4 of Example 6-11. The spreadsheet formulas give the same answer as shown in Figure 6-1.

\[
\text{interest}_4 = \text{IPMT}(0.5\%,4,6,-2400) = 6.04 \\
\text{principal payment}_4 = \text{PPMT}(0.5\%,4,6,-2400) = 400.98
\]

Finding the Balance Due on a Loan

An amortization schedule can be used to calculate the balance due on a loan. Or more easily the balance due equals the present worth of the remaining payments. Interest is
paid in full after each payment, so later payments are simply based on the balance due.

**EXAMPLE 6-12**
A car is purchased with a 48-month, 9% nominal loan with an initial balance of $15,000. What is the balance due half-way through the four years?

**Solution**
The first step is to calculate the monthly payment, at a monthly interest rate of ¾%. This equals

\[
payment = 15,000 \cdot (A/P, 0.75\%, 48) \text{ or } = \text{PMT}(0.75\%, 48, -15000) \\
= 15,000 \cdot 0.0249 = $373.50 \\
= $373.28
\]

The next step will use the spreadsheet answer, because it is more accurate (there are only three significant digits in the tabulated factor).

After 24 payments and with 24 left, the remaining balance equals

\[
(P/A, i, N_{\text{remaining}}) \cdot \text{payment} \\
\text{balance} = (P/A, 0.75\%, 24) \cdot $373.28 \text{ or } = \text{PV}(0.75\%, 24, 373.28) \\
= 21.889 \cdot 373.28 = $8170.73 \\
= $8170.78
\]

Thus half-way through the repayment schedule 54.5% of the original balance is still owed.

---

**Payoff Debt Sooner by Increasing Payments**

Paying off debt can be a good investment, as the investment earns the rate of interest on the loan. For example, this could be 8% for a mortgage, 10% for a car loan, or 19% for a credit card. When making extra payments on a loan, the common question is: How much sooner will the debt be paid off? Until the debt is paid off, any early payments are essentially locked up, because the same payment amount is owed each month.

The first reason that spreadsheets are convenient is fractional interest rates. For example, an auto loan might be at a nominal rate of 13% with monthly compounding or 1.08333% per month. The second reason is that NPER calculates the number of periods remaining on a loan.

NPER can be used to calculate “How much difference does one extra payment make?” or “How much difference does increasing all payments by x% make?” Extra payments are applied entirely to principal, so the interest rate, remaining balance, and payment amounts are all known. \(N_{\text{remaining}}\) equals NPER(\(i\), payment, remaining balance) with optional arguments for beginning of period cash flows and balloon payments. The signs of the payment and the remaining balance must be different.
EXAMPLE 6-13

Maria has a 7.5% mortgage with monthly payments for 30 years. Her original balance was $100,000, and she just made her 12th payment. Each month she also pays into a reserve account, which the bank uses to pay her fire and liability insurance ($900 annually) and property taxes ($1500 annually). How much does she shorten the loan by, if she makes an extra loan payment today? If she makes an extra total payment? If she increases each total payment to 110% of her current total payment?

Solution
The first step is to calculate her loan payment for the 360 months. Rather than calculating a six-significant digit monthly interest rate, it is easier to use 0.075/12 in the spreadsheet formulas.

\[ \text{payment} = \text{PMT}(0.075/12, 360, -100000) = \$699.21 \]

The remaining balance after 12 such payments is the present worth of the remaining 348 payments.

\[ \text{balance}_{12} = \text{PV}(0.075/12, 348, 699.21) = \$99,077.53 \]

(after 12 payments she has paid off $922!)

If she pays an extra $699.21, then the number of periods remaining is

\[ \text{NPER}(0.075/12, -699.21, 99077.53-699.21) = 339.5 \]

This is 8.5 payments less than the 348 periods left before the extra payment.

If she makes an extra total payment, then

\[ \text{total payment} = 699.21 + 900/12 + 1500/12 = \$899.21 / \text{month} \]

\[ \text{NPER}(0.075/12, -899.21, 99244 - 899.21) = 337.1 \text{ or 2.4 more payments saved.} \]

If she makes an extra 10% payment on the total payment of $899.21, then

\[ \text{NPER}(0.075/12, -(1.1*899.21-200), 99077.53) = 246.5 \text{ payments or 101.5 payments saved.} \]

Note that $200 of the total payment goes to pay for insurance and taxes.□
**Summary**

*Annual cash flow analysis* is the second of the three major methods of resolving alternatives into comparable values. When an alternative has an initial cost $P$ and salvage value $S$, there are three ways of computing the equivalent uniform annual cost:

- $EUAC = P(A/P,i,n) - S(A/F,i,n)$
- $EUAC = (P - S)(A/F,i,n) + Pi$
- $EUAC = (P - S)(A/P,i,n) + Si$

All three equations give the same answer.

The relationship between the present worth of cost and the equivalent uniform annual cost is:

- $EUAC = (PW \text{ of cost})(A/P,i,n)$

The three annual cash flow criteria are:

- For fixed input
  - Maximize $EUAB$
- For fixed output
  - Minimize $EUAC$
- Neither input nor output fixed
  - Maximize $(EUAB - EUAC)$

In present worth analysis there must be a common analysis period. Annual cash flow analysis, however, allows some flexibility provided the necessary assumptions are suitable in the situation being studied. The analysis period may be different from the lives of the alternatives, and a valid cash flow analysis made, provided the following two criteria are met:

1. When an alternative has reached the end of its useful life, it is assumed to be replaced by an identical replacement (with the same costs, performance, and so forth).

2. The analysis period is a common multiple of the useful lives of the alternatives, or there is a continuing or perpetual requirement for the selected alternative.

If both these conditions do not apply, then it is necessary to make a detailed study of the consequences of the various alternatives over the entire analysis period with particular attention to the difference between the alternatives at the end of the analysis period.

There is very little numerical difference between a long-life alternative and a perpetual alternative. As the value of $n$ increases, the capital recovery factor approaches $i$. At the limit, $(A/P,i,\infty) = i$. 