Initial and Final Value Theorems

Final Value Theorem - determines the steady-state value of the system response without finding the inverse transform.

Procedure:

1.) find the transfer function X(s)
2.) multiply X(s) by s
3.) take the limit of sX(s) as S goes to zero
4.) result is value of x(t) when t = infinity

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s \cdot X(s)$$

Initial Value Theorem - determines the value of the time function when t=0 without finding the inverse transform

Procedure:

1.) find the transfer function X(s)
2.) multiply X(s) by s
3.) take the limit of sX(s) as S goes to infinity
4.) result is value of x(t) when t =0

$$\lim_{t \to 0} x(t) = \lim_{s \to \infty} s \cdot X(s)$$

Example 1: Find the initial value of the transfer function

$$X(s) = \frac{808}{s\cdot\left(s^2 + 2\cdot s + 101\right)}$$

$$x(0) = \lim_{t\to 0} x(t) = \lim_{s\to\infty} s\,X(s) = \lim_{s\to\infty} \frac{s\,808}{s(s^2+2s+101)}$$

$$\lim_{s\to\infty} = \frac{808}{s^2+2s+101}$$ Larger values of s make denominator larger ratio → 0

$$x(0) = \lim_{s\to\infty} \frac{808}{s^2+2s+101} = \underline{\underline{0}}$$ ANS

Example 2: Find the final value of the transfer function X(s) above.

Fina Value $\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\,X(s)$

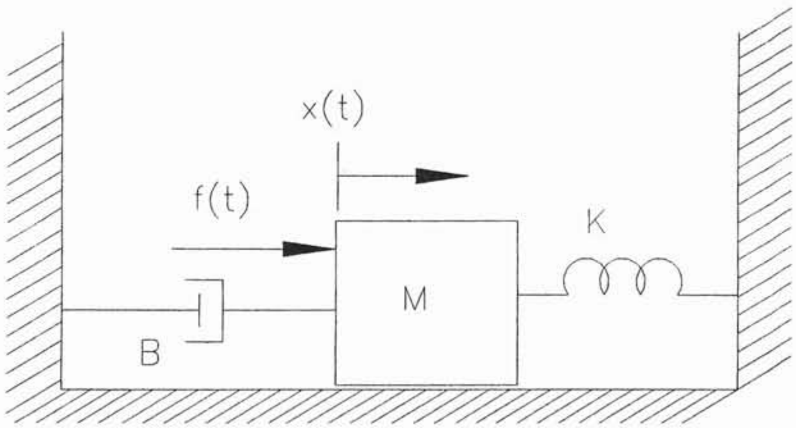
$$\lim_{s\to 0} s\,X(s) = \lim_{s\to 0} \frac{s(808)}{s(s^2+2s+101)} = \lim_{s\to 0} \frac{808}{s^2+2s+101}$$

as $s \to 0$ $s^2 \to 0$ and $2s \to 0$

$$\lim_{s\to 0} \frac{808}{s^2+2s+101} = \frac{808}{101} = \underline{\underline{8}}$$

The final value $\underline{\underline{x(\infty) = 8}}$

Using Laplace for solving mechanical systems



Write the differential equations for the above system with respect to position and solve them using Laplace transform methods. Assume f(t) = F and that the mass slides on a frictionless surface. x(0)=0

$$f(t) = M\cdot\frac{d^2}{dt^2}x(t) + B\cdot\frac{d}{dt}x(t) + K\cdot x(t)$$

$$F = M\cdot\frac{d^2}{dt^2}x(t) + B\cdot\frac{d}{dt}x(t) + K\cdot x(t)$$

Take Laplace transform of both sides

$$\frac{F}{s} = M\cdot s^2\cdot X(s) + B\cdot s\cdot X(s) + K\cdot X(s)$$

Solve for the position X(s)

$$\frac{F}{s\left(M\cdot s^2 + B\cdot s + K\right)} = X(s)$$

Let M = 1, F = 5, B = 4 and K = 5. Solve this using Laplace and partial fraction expansion.

$$\frac{5}{s\left(1\cdot s^2 + 4\cdot s + 5\right)} = X(s)$$

Use quadratic formula to factor denominator

$a := 1 \quad b := 4 \quad c := 5$

$$s_1 := \frac{-b + \sqrt{b^2 - 4\cdot a\cdot c}}{2\cdot a} \qquad s_2 := \frac{-b - \sqrt{b^2 - 4\cdot a\cdot c}}{2\cdot a}$$

$s_1 = -2 + i \qquad s_2 = -2 - i$

Factored form of function

$$\frac{5}{s\cdot(s + (2 + j))\cdot(s + (2 - j))} = X(s)$$

Use partial fraction expansion

$$\frac{5}{s(s+(2+j))(s+(2-j))} = \frac{A}{s} + \frac{B}{s+(s+j)} + \frac{C}{s+(2-j)}$$

Find A multiply by s

$$\left.\frac{5s}{s(s+(2+j))(s+(2-j))}\right|_{s=0} = \left.\frac{As}{s} + \frac{Bs}{s+(2+j)} + \frac{Cs}{s+(2-j)}\right|_{s=0}$$

$$\frac{5}{(2+j)(2-j)} = A = \frac{5}{2^2+1} = \frac{1}{5}$$

Mechancical system solution: Continued

Solve For B

$$\left.\frac{5(s+(2+j))}{s(s+(2+j))(s+(2-j))}\right| = \frac{A(s+2+j)}{s} + \frac{B(s+(2+j))}{s+(2+j)} + \frac{C(s+(2+j))}{s+(2-j)}\Bigg|_{s=-(2+j)}$$

$s=-(2+j)$

$$\frac{5}{(-2-j)(-2-j+2-j)} = \frac{A(-(2+j)+(2+j))^{\nearrow 0}}{s} + B + \frac{C(-(2+j)+(2+j))^{\nearrow 0}}{s+(2-j)}$$

$$\frac{5}{(-2-j)(-2j)} = B = \frac{5}{4j-2}$$

Solve for C

$$\left.\frac{5(s+(2-j))}{s(s+(2+j))(s+(2-j))}\right| = \frac{A(s+(2-j))}{s} + \frac{B(s+(2-j))}{s+(2+j)} + \frac{C(s+(2-j))}{s+(2-j)}\Bigg|_{s=-(2-j)}$$

$s=-(2-j)$

$$\left.\frac{5}{s(s+(2+j))}\right|_{s=-(2-j)} = A\frac{(-(2-j)+(2-j))^{\nearrow 0}}{-(2-j)} + \frac{B(-(2-j)+(2-j))^{\nearrow 0}}{-(2-j)+(2+j)} + C$$

$$\frac{5}{(-2+j)(-2+j+2+j)} = C = \frac{5}{(-2+j)(2j)} = \frac{5}{-4j-2} = C$$

B & C complex conjugates

$$\mathcal{L}^{-1}\left[\frac{5}{s(s+(2-j))(s+(2+j))}\right] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] + \left(\frac{5}{-2+4j}\right)\mathcal{L}^{-1}\left[\frac{1}{s+(2+j)}\right] +$$

$$\rightarrow \left[\frac{5}{-2-4j}\right]\mathcal{L}^{-1}\left[\frac{1}{s+(2-j)}\right]$$

Mechanical Solution

$$X(t) = 1 + \left[\frac{5}{-2+4j}\right] e^{-(2+j)t} + \left[\frac{5}{-2-4j}\right] e^{-(2-j)t}$$

$$X(t) = 1 + \left[\frac{5}{-2+4j}\right] e^{-2t} e^{-jt} + \left[\frac{5}{-2-4j}\right] e^{-2t} e^{+jt}$$

Rationalized denominators

$$\left(\frac{5}{-2+4j}\right)\left(\frac{-2-4j}{-2-4j}\right) = \frac{-10-20j}{4+16} = -1/2 - j$$

$$\left[\frac{5}{-2-4j}\right]\left[\frac{-2+4j}{-2+4j}\right] = \frac{-10+20j}{4+16} = -1/2 + j$$

$$X(t) = 1 + e^{-2t}\left[(-1/2 - j)e^{-jt} + (-1/2 + j)e^{jt}\right]$$

$$X(t) = 1 + e^{-2t}\left[-1/2\, e^{-jt} - 1/2\, e^{jt} - j e^{-jt} + j e^{jt}\right]$$

$$X(t) = 1 + e^{-2t}\left[-1/2\left(e^{-jt} + e^{jt}\right) + j\left(e^{jt} - e^{-jt}\right)\right]$$

$e^{-jt} + e^{jt} = 2\cos t$ Euler's & $e^{jt} - e^{-jt} = 2j\sin t$ Euler's

$$X(t) = 1 + e^{-2t}\left[-\cos t - 2\sin(t)\right]$$

$$X(t) = 1 - e^{-2t}\left[\cos(t) + 2\sin(t)\right] \text{ Finally}$$

once again Mathcad can do most of this work

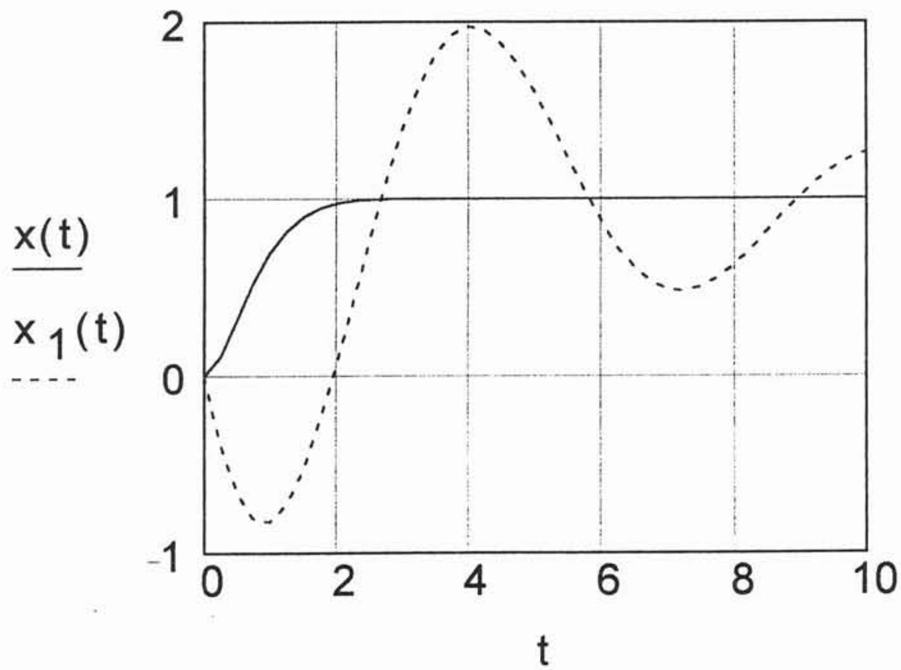
Plot of the mechanical system response

$$x(t) := 1 - e^{-2\cdot t}\cdot(\cos(t) + 2\cdot\sin(t))$$

-2 relates to damping of system decrease and see effects

$$x_1(t) := 1 - e^{-.2\cdot t}\cdot(\cos(t) + 2\cdot\sin(t))$$

$$t := 0, 0.25 .. 10$$



Transfer Functions

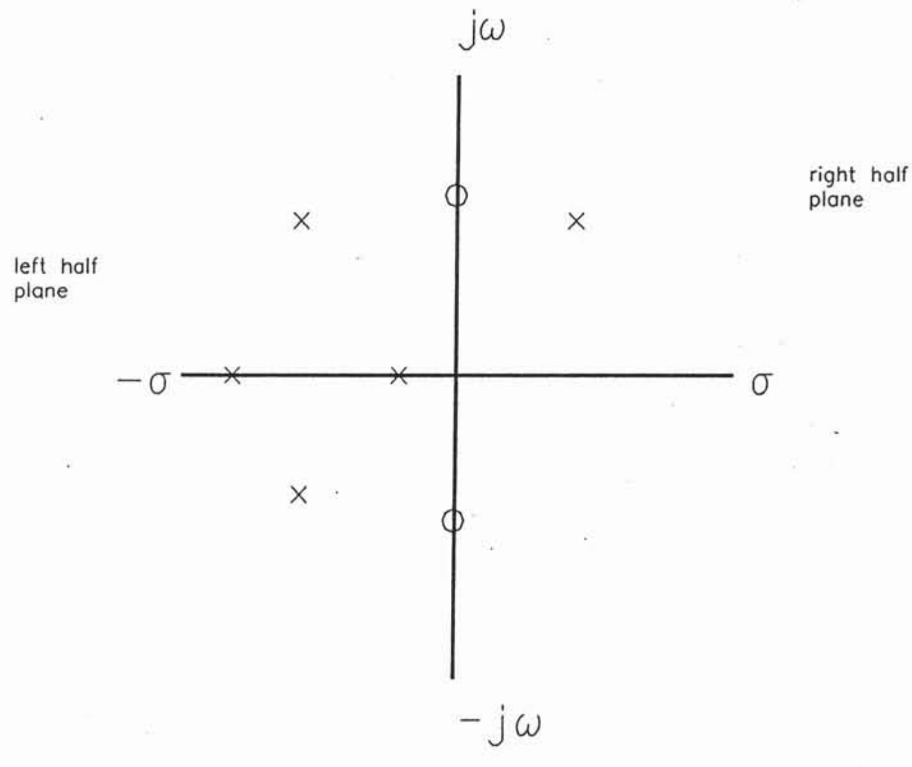
Input/output relationships for a mathematical model usually given by the ratio of two polynominals of the variable s

Definitions

Poles - roots of the denominator polynomial. Values that cause transfer function magnitude to go to infinity.

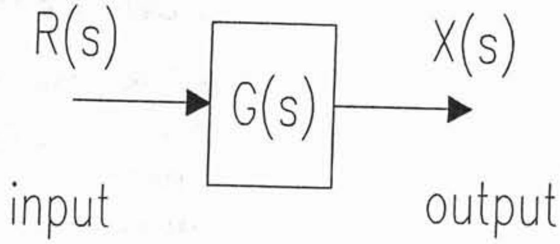
Zeros - roots of the numerator polynomial. Values that cause the transfer function to go to 0.

eigenvalues - Characteristic responses of a system. Roots of the denominator polynomial. All eigenvalues must be negative for a system transient (natural response) to decay out.



X's indicate location of pole. 0 is location of zero
Closer pole is to imaginary axis slower response. Complex roots appear in conjugate pairs

Examples

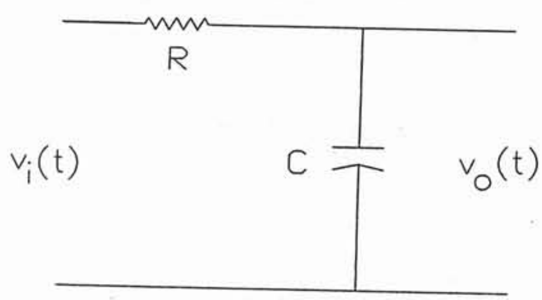


Transfer Function is a "gain" as a function of s

$$X(s) = G(s) \cdot R(s)$$

$$\frac{X(s)}{R(s)} = G(s)$$

Passive Lowpass filter - intergrator



$$V_i(t) = R \cdot i(t) + \frac{1}{C} \cdot \int i(t)\, dt$$

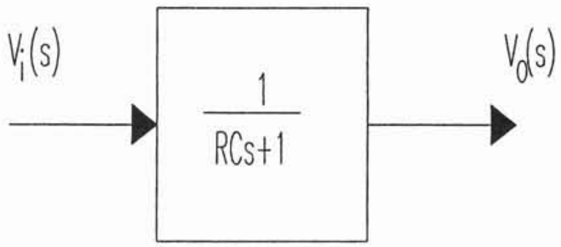
Take Laplace

$$V_i(s) = R \cdot I(s) + \frac{1}{C \cdot s} \cdot I(s)$$

$$\frac{V_i(s)}{R + \frac{1}{C \cdot s}} = I(s)$$

Remember $V_o(s) = \frac{1}{C \cdot s} \cdot I(s)$

So $$V_o(s) = \frac{\frac{1}{C \cdot s}}{R + \frac{1}{C \cdot s}} \cdot V_i(s) = \frac{1}{R \cdot C \cdot s + 1} \cdot V_i(s)$$ Voltage divider formula

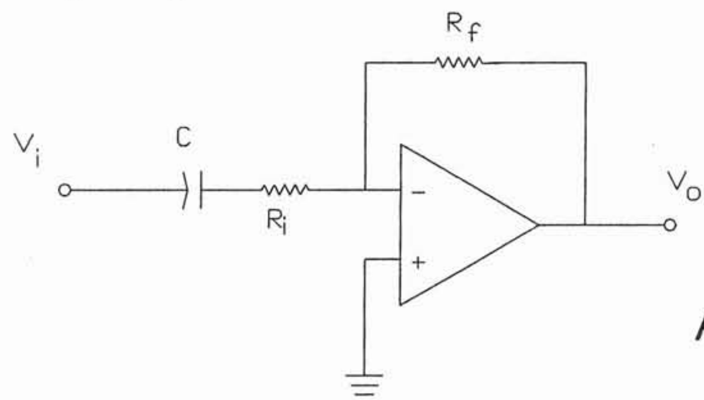


Draw as a block diagram

RC is time constant of system. System has 1 pole at -1/RC and no zeros
Larger RC slower response

Transfer functions of OP AMP circuits

Practical Differentiator- active high pass filter with difinite low frequency cutoff.



Take Laplace of components and treat like impedances

General gain formula

$$A_v(s) = \frac{-z_f(s)}{z_i(s)} = \frac{V_o(s)}{V_i(s)}$$

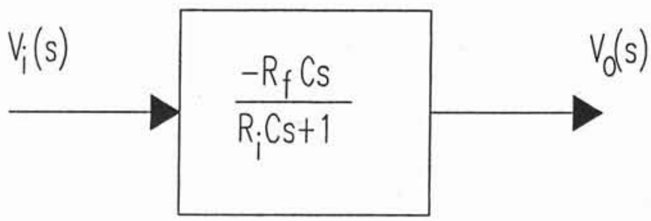
$$z_i(s) = R_i + \frac{1}{C \cdot s} \qquad z_f(s) = R_f \qquad A_v(s) = \frac{-R_f}{R_i + \frac{1}{C \cdot s}}$$

Simplify A_v

$$A_v(s) = \frac{-R_f \cdot C \cdot s}{R_i \cdot C \cdot s + 1} = \frac{V_o(s)}{V_i(s)}$$

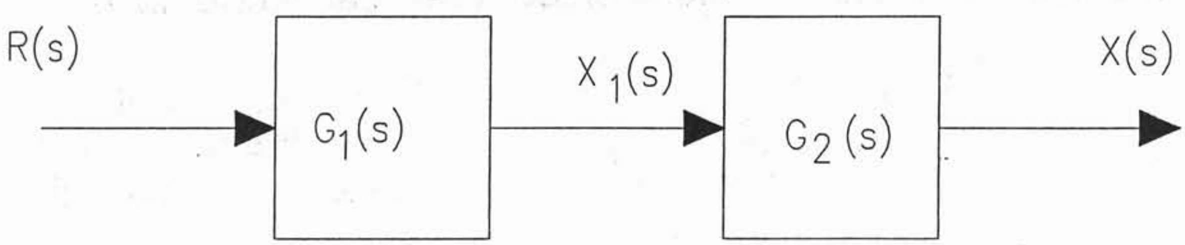
Transfer function has 1 zero at s=0 and 1 pole at $s = -1/R_iC$

Transfer function for Practical Differentiator



Block Algebra for transfer functions- Cascaded blocks

Series connected - multiply transfer functions
Note: do not cancel common terms from numerator and denominator

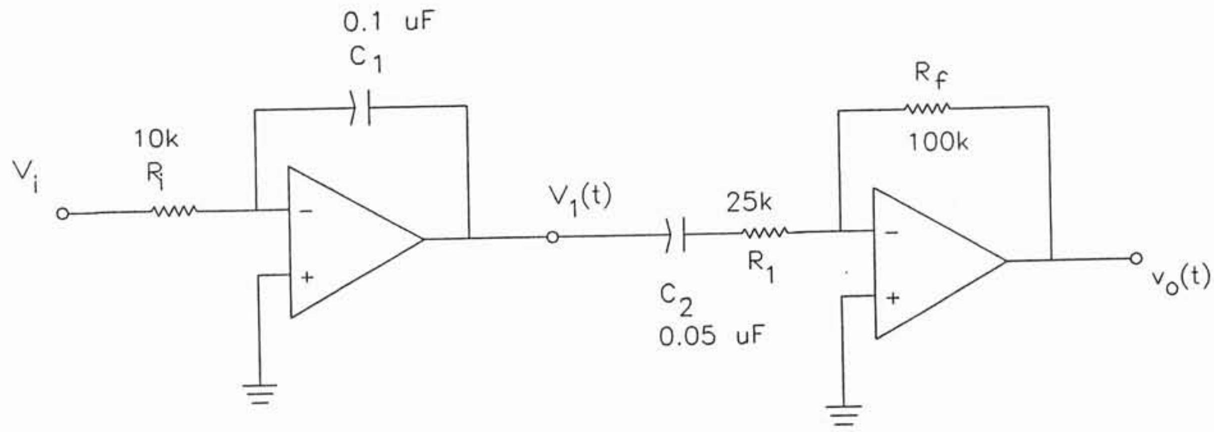


$X_1(s) = R(s) \cdot G_1(s)$ (1) $\quad$ $X(s) = G_2(s) \cdot X_1(s)$ (2)

$X(s) = G_1(s) \cdot G_2(s) \cdot R(s)$ $\quad$ $\frac{X(s)}{R(s)} = G_1(s) \cdot G_2(s)$

Substitute (1) into (2) and simplify to get overall gain

Example with OP AMPs



First stage- integrator Second stage- practical differentiator

$$G_1(s) = \frac{V_1(s)}{V_i(s)} \qquad G_2(s) = \frac{V_o(s)}{V_1(s)}$$

Take Laplace of components and use general gain formula

For stage 1

$$A_{v1}(s) = \frac{-z_f(s)}{z_i(s)} \quad z_i(s) = R_i \quad z_f(s) = \frac{1}{C_1 \cdot s} \quad A_{v1}(s) = \frac{\frac{-1}{C_1 \cdot s}}{R_i}$$

Simplify A_{v1} to get $G_1(s)$

$$G_1(s) = \frac{-1}{R_i \cdot C_1 \cdot s}$$

$G_2(s)$ from previous example

$$A_v(s) = \frac{-R_f \cdot C_2 \cdot s}{R_1 \cdot C_2 \cdot s + 1} = G_2(s)$$

$$\frac{V_o(s)}{V_i(s)} = \frac{-1}{R_i \cdot C_1 \cdot s} \cdot \frac{-R_f \cdot C_2 \cdot s}{R_1 \cdot C_2 \cdot s + 1}$$

Negative signs cancel

$$\frac{V_o(s)}{V_i(s)} = \frac{R_f \cdot C_2 \cdot s}{(R_1 \cdot C_2 \cdot s + 1) \cdot (R_i \cdot C_1 \cdot s)}$$ Simplified form

Plug in given values for the component symbols and compute parameters

$R_i := 10000$ $C_1 := 0.1 \cdot 10^{-6}$ $C_2 := 0.05 \cdot 10^{-6}$

$R_1 := 25000$ $R_f := 100000$ $R_1 \cdot C_2 = 0.001$

$R_i \cdot C_1 = 0.001$ $R_f \cdot C_2 = 0.005$

Above are all time constants for the system

Final transfer function

$$\frac{V_o(s)}{V_i(s)} = \frac{0.005 \cdot s}{(0.001 \cdot s + 1) \cdot (0.001 \cdot s)}$$

Function has 1 zero at s = 0 and two poles s = -1/0.001 = 1000 and s = 0

Parallel Blocks - add transfer functions