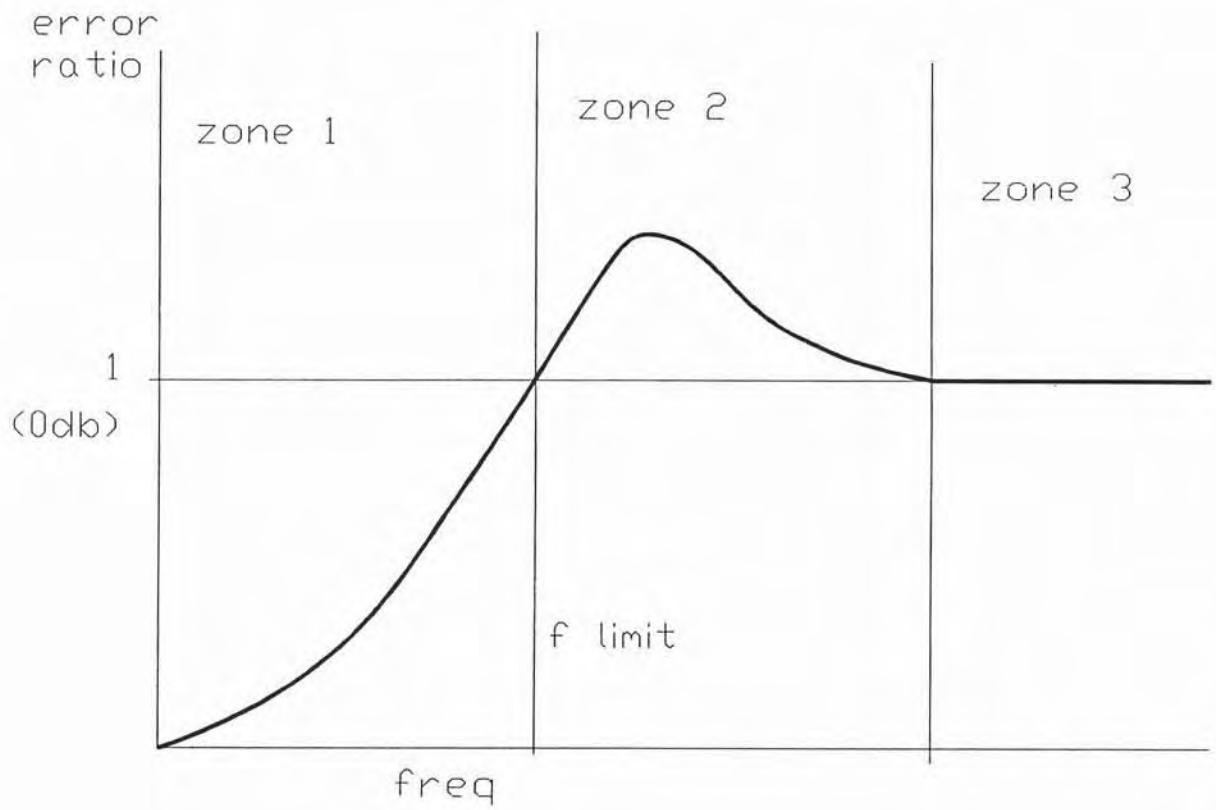
The transition between Zones 1 and 2 is the frequency limit of a control system.



Error Ratio (ER) = ratio of closed-loop error magnitude to open-loop error magnitude

$$ER = \left| \frac{1}{(1 + G(s) \cdot H(s)) \cdot (1 - G(s) \cdot H(s))} \right|$$

Magnitude of the complex number given by the above

Example: Find: 1.) the open loop transfer function, 2.) the closed loop transfer function, 3.) the error ratio of a control system where the forward gain G(s) is:

$$G(s) := \frac{21.8}{1 + 0.379 \cdot s + 0.0063 \cdot s^2}$$

The feedback system gain is:

$$H(s) := \frac{.356}{1 + 0.478 \cdot s}$$

4.) Compute value of the open and closed loop transfer functions when ω = 0.1, 1, 10 and 100.
5.) Compute the value of the error ratio when ω = 0.1 1, 10 and 100.
6.) Use Matlab to plot the open and closed loop transfer functions on the same Bode plot axis.

1.) OPEN LOOP TRANSFER FUNCTION

$$G(s)H(s) = \frac{21.8}{1+0.379s+0.0063s^2}\left[\frac{0.356}{1+0.478s}\right]$$

Multiply denominator

$$(1+0.379s+0.0063s^2)(1+0.478s) = 1+0.379s+0.478s+0.18116s^2+0.0063s^2+3.01\times10^{-3}s^3$$

$$\boxed{G(s)H(s) = \frac{7.7608}{1+0.857s+0.187465s^2+0.003014s^3}}$$

ANS

Example (cont.) 2.) CLOSED LOOP TRANSFER FUNCTION

$$\frac{G(s)H(s)}{1+G(s)H(s)} = \frac{\dfrac{7.7608}{1+0.857s+0.18746s^2+0.003015s^3}}{1+\dfrac{7.7608}{1+0.857s+0.18746s^2+0.003015s^3}}$$

$$\frac{G(s)H(s)}{1+G(s)H(s)} = \frac{7.7608}{(1+0.857s+0.18746s^2+0.003015s^3)+7.7608}$$

$$\frac{G(s)H(s)}{1+G(s)H(s)} = \boxed{\frac{7.7608}{8.7608+0.857s+0.18746s^2+0.003015s^3}}$$ ANS

3.) ERROR RATIO

MAGNITUDE ONLY
expand denominator to simplify

$$ER(s) = \left|\frac{1}{1+G(s)H(s)}\left(\frac{1}{1-G(s)H(s)}\right)\right|$$

$$ER(s) = \left|\frac{1}{(1+G(s)H(s)-G(s)H(s)-[G(s)H(s)]^2}\right| = \left|\frac{1}{1-(G(s)H(s))^2}\right|$$

SUBSTITUTE IN $G(s)H(s)$ FROM ABOVE

$$ER(s) = \left|\frac{1}{1-\left(\dfrac{7.7608}{1+0.857s+0.18746s^2+0.003015s^3}\right)^2}\right|$$

$$ER(s) = \left|\frac{(1+0.857s+0.18746s^2+0.003015s^3)^2}{(1+0.857s+0.18746s^2+0.003015s^3)^2-(7.7608)^2}\right|$$

$$\boxed{ER(s) = \left|\frac{(1+0.857s+0.18746s^2+0.003015s^3)^2}{(1+0.857s+0.18746s^2+0.003015s^3)^2-60.23}\right|}$$ ANS

Example (cont.)

4.) Compute the values of the open and closed loop transfer functions at ω= 0.1, 1, 10, 100 rad/sec

OPEN LOOP

$$G(\omega)\cdot H(\omega) = \frac{7.7608}{1 + 0.857\cdot j\cdot\omega + 0.1875\cdot(j\cdot\omega)^2 + 0.0030\cdot(j\cdot\omega)^3}$$

$$G(\omega)H(\omega) = \frac{7.7608}{1 + 0.857j\omega - 0.1875\omega^2 - 0.003j\omega^3} = GH(\omega)$$

$$\begin{bmatrix} j^2 = -1 \\ j^3 = -j \end{bmatrix}$$

Substitute in $\omega = 0.1$ and simplify

$$GH(0.1) = \frac{7.7608}{1 + 0.857j(0.1) - 0.1875(0.1)^2 - 0.003j(0.1)^3}$$

$$GH(0.1) = \frac{7.7608}{1 + 0.001875 + 0.0857j - 3\times10^{-6}j}$$

$$GH(0.1) = \frac{7.7608}{0.99813 + j0.085697} = \frac{7.7608}{1.002\angle 4.91^\circ}$$

$$\boxed{GH(0.1) = 7.746\angle -4.91^\circ}$$ ANS

$$GH(1) = \frac{7.7608}{1 + 0.857j(1) - 0.1875(1)^2 - 0.003j(1)^3}$$

$$GH(1) = \frac{7.7608}{1 - 0.1875 + 0.857j - 0.003j} = \frac{7.7608}{0.8125 + j0.854}$$

$$GH(1) = \frac{7.7608}{1.1788\angle 46.42} = \boxed{6.584\angle -46.4^\circ}$$

Example: (cont.)

OPEN LOOP $GH(10) = \dfrac{7.7608}{1+0.857j(10)-0.1875(10)^2-0.003j(10)^3}$

$$GH(10) = \frac{7.7608}{1-18.75+8.57j-3j} = \frac{7.7608}{-17.75+5.57j}$$

$$GH(10) = \frac{7.7608}{18.603\angle 162.57^\circ} = \boxed{0.4172\angle -162.6^\circ}$$

$$GH(100) = \frac{7.7608}{1+0.857j100-0.1875(100)^2-0.003j(100)^3}$$

$$GH(100) = \frac{7.7608}{1+85.7j-1875-3000j} = \frac{7.7608}{-1874-j2914.3}$$

$$GH(100) = \frac{7.7608}{3464.83\angle -122.74} = \boxed{0.00224\angle 122.74^\circ}$$

NOTE Phase Changes

CONVERT TO db

$20 \,\text{Log}\,|GH(0.1)| = 20\,\text{Log}(7.746) = 17.782\text{ db}$ -4.91° phase

$20\,\text{Log}\,|GH(1.0)| = 20\,\text{Log}(6.584) = 16.370\text{ db}$ -46.4° phase

$20\,\text{Log}\,|GH(10)| = 20\,\text{Log}(0.4172) = -7.593\text{ db}$ -162.6° phase

$20\,\text{Log}\,|GH(100)| = 20\,\text{Log}(0.00224) = -53.0\text{ db}$ $+122.6^\circ$ phase

COMPUTE CLOSED LOOP TRANSFER FUNCTION VALUES IN SAME FASHION.

Computing values of closed loop transfer function.

Make this definition $\frac{G(s)\cdot H(s)}{1 + G(s)\cdot H(s)} = GH_c(s)$

$$GH_c(\omega) = \frac{7.7608}{8.7608 + 0.857j\omega + 0.1875(j\omega)^2 + 0.003(j\omega)^3}$$

$$GH_c(\omega) = \frac{7.7608}{8.7608 + 0.857j\omega - 0.1875\omega^2 - 0.003j\omega^3}$$

$$GH_c(0.1) = \frac{7.7608}{8.7608 + 0.857j(0.1) - 0.1875(0.1)^2 - 0.003j(0.1)^3} = 0.886 - 8.67\times10^{-3}j$$

$$GH_c(0.1) = \boxed{0.886\angle -0.56^\circ}$$

$$GH_c(1) = \frac{7.7608}{8.7608 + 0.857j(1) - 0.1875(1)^2 - 0.003j(1)^3} = 0.896 - 0.089j$$

$$GH_c(1) = \boxed{0.9004\angle -5.67^\circ}$$

$$GH_c(10) = \frac{7.7608}{8.7608 + 0.857j(10) - 0.1875(10)^2 - 0.003j(10)^3} = -0.593 - 0.33j$$

$$GH_c(10) = \boxed{0.6786\angle -150.9^\circ}$$

$$GH_c(100) = \frac{7.7608}{8.7608 + 0.857j(100) - 0.1875(100)^2 - 0.003j(100)^3} = -1.21\times10^{-3} + 1.89\times10^{-3}j$$

$$GH_c(100) = \boxed{0.002244\angle 122.62^\circ}$$

CONVERT TO db

$20\,Log|GH_c(0.1)| = 20\,Log(0.886) = -1.051\,db \quad -0.56^\circ$ Phase

$20\,Log|GH_c(1)| = 20\,Log(0.9004) = -0.911\,db \quad -5.67^\circ$ Phase

Example (cont.) $20 \log|GH_c(10)| = 20 \log(0.6786) = -3.368\,db$

phase -150.9°

$20 \log|GH_c(100)| = 20 \log(0.002244) = -52.98\,db \quad 122.62^\circ$ phase

5.) Compute ERROR RATIO for $\omega = 0.1, 1, 10, 100$ rad/sec

Definition of error ratio

$$ER = \left|\frac{1}{(1+G(s)H(s))(1-G(s)H(s))}\right| = \left|\frac{1}{(1+GH(s))(1-GH(s))}\right|$$

AT $\omega = 0.1$ $GH(0.1) = 7.746\angle -4.91^\circ$

$$ER = \left|\frac{1}{(1+7.746\angle -4.91^\circ)(1-7.746\angle -4.91^\circ)}\right|$$

$$ER = \left|\frac{1}{59.015\angle 170.0^\circ}\right| = \boxed{0.017}$$

AT $\omega = 1$ $GH(1) = 6.584\angle -46.4$

$$ER = \left|\frac{1}{(1+6.584\angle -46.4^\circ)(1-6.584\angle -46.4^\circ)}\right|$$

$$ER = \left|\frac{1}{43.409\angle 85.9^\circ}\right| = \boxed{0.023}$$

AT $\omega = 10$ $GH(10) = 0.4172\angle -162.6^\circ$

$$ER = \left|\frac{1}{(1+0.4172\angle -162.6^\circ)(1-0.4172\angle -162.6^\circ)}\right|$$

$$ER = \left|\frac{1}{0.8628\angle -6.61^\circ}\right| = \boxed{1.159}$$

Example (cont.)
Complete the computations of the error ratio

AT $\omega=100$ $GH(100) = 0.00224\angle 122.74^\circ$

$$ER = \left|\frac{1}{(1+0.00224\angle 122.74^\circ)(1-0.00224\angle 122.74^\circ)}\right| = \left|\frac{1}{1\angle 2.62^\circ}\right| = \boxed{1}$$

convert to db values

$\omega=0.1$ $20\,Log(0.017) = -35.39$ db

$\omega=1$ $20\,Log(0.023) = -32.77$ db

$\omega=10$ $20\,Log(1.159) = 1.28$ db

$\omega=100$ $20\,Log(1) = 0$ db

As the frequency increases the error ratio increase. It peaks and then comes to a constant value of 1.

The three zones are defined as:

Zone 1: error ratio < 0 db; good control; controller decreases the error
Zone 2: error ratio > 0 db poor control; controller increase the error
Zone 3: error ration = 0 db no control; controller does not affect the error

The Matlab code to generate the bode plots follows

```
% Example bode calculations
clear all;
close all;
% define the forward gain numerator and denominator coefficients
numg=[21.8];
demg=[0.0063 0.379 1];
% define the feedback path gain numerator and denominators
numh=[0.356];
demh=[0.478 1];
% construct the transfer functions
G=tf(numg,demg);
H=tf(numh,demh);
% find GH(s)
GH=G*H
% find the closed loop transfer function
GHc=GH/(1+GH)
% compare it to the derived closed loop transfer function
% note that the program does not give the minmum algebraic model
GHt=tf([7.7608],[0.00301 0.1875 0.857 8.7608])
% this statement plots all three systems on the same axis
% the open loop system is in green with o's at every data point
% the Matlab derived closed loop functoin is in red
% and the closed loop function derived by hand is in blue with
% o's at every data point. The value in curly brackets are freq. limits
bode(GH,'go-',GHc,'r-',GHt,'bo',{0.1,100})
```

Transfer function: G(S)H(S)

```
            7.761
---------------------------------------------------
0.003011 s^3 + 0.1875 s^2 + 0.857 s + 1
```

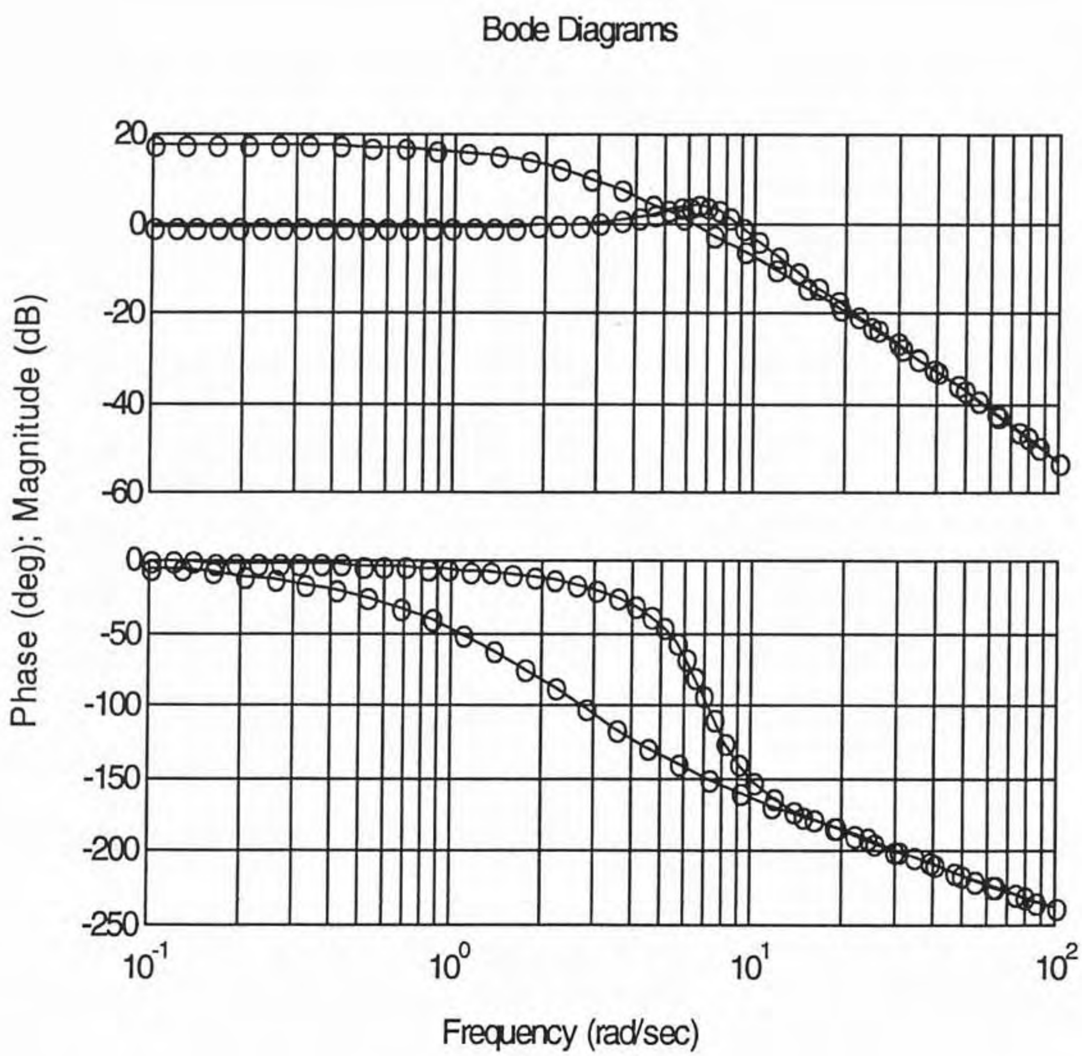
Transfer function: G(s)H(s)/(1+G(s)H(s)) machine generated

```
            0.02337 s^3 + 1.455 s^2 + 6.651 s + 7.761
---------------------------------------------------------------------------
9.069e-006 s^6 + 0.001129 s^5 + 0.0403 s^4 + 0.3507 s^3 +
2.564 s^2 + 8.365 s + 8.761
```

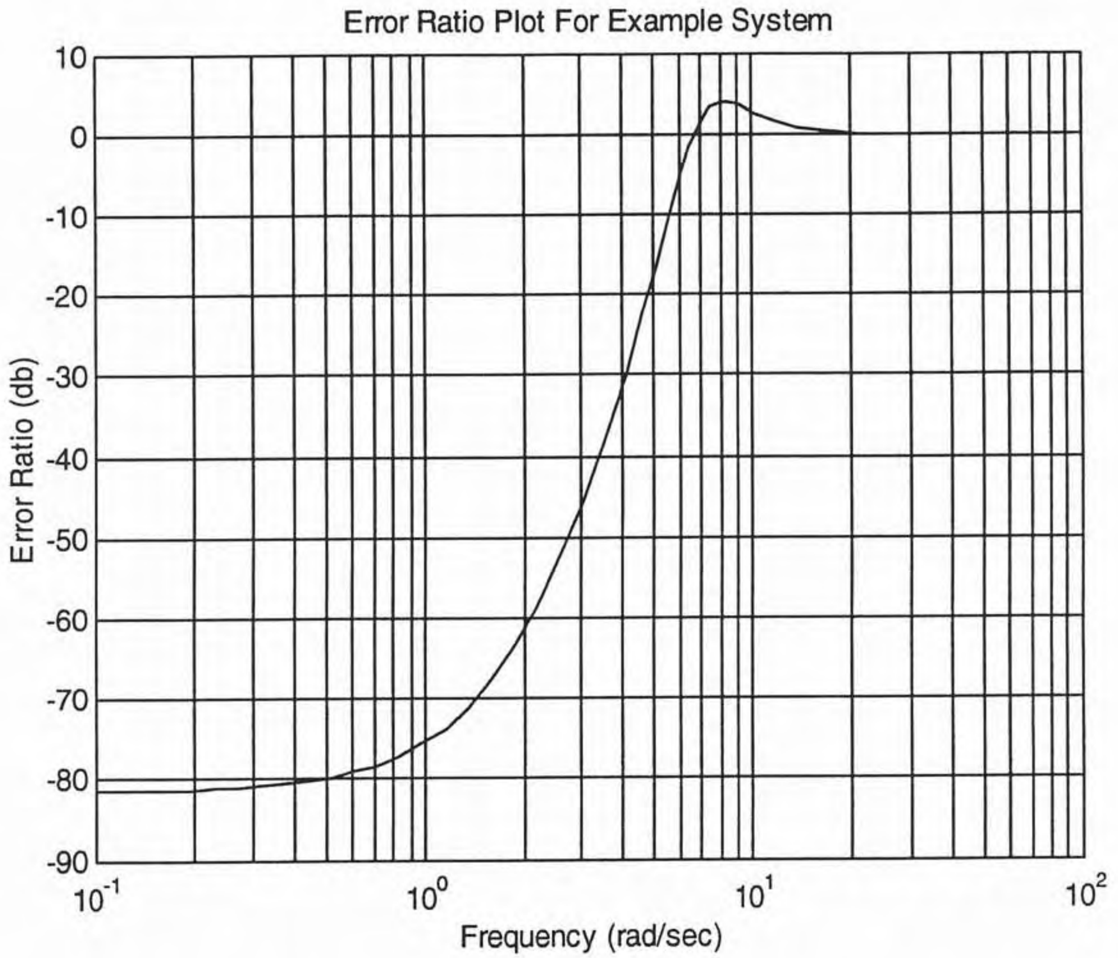
The last transfer function can be factored and simplified into the same form as the hand-derived transfer function

Transfer function: G(s)H(s)/(1+G(s)H(s)) hand derived

```
              7.761
----------------------------------------------------
0.00301 s^3 + 0.1875 s^2 + 0.857 s + 8.761
```



Plot the error ratio for the example system and determine the Maximum frequency limit.



Maximum frequency limit occurs when the error ratio first crosses 0 db.

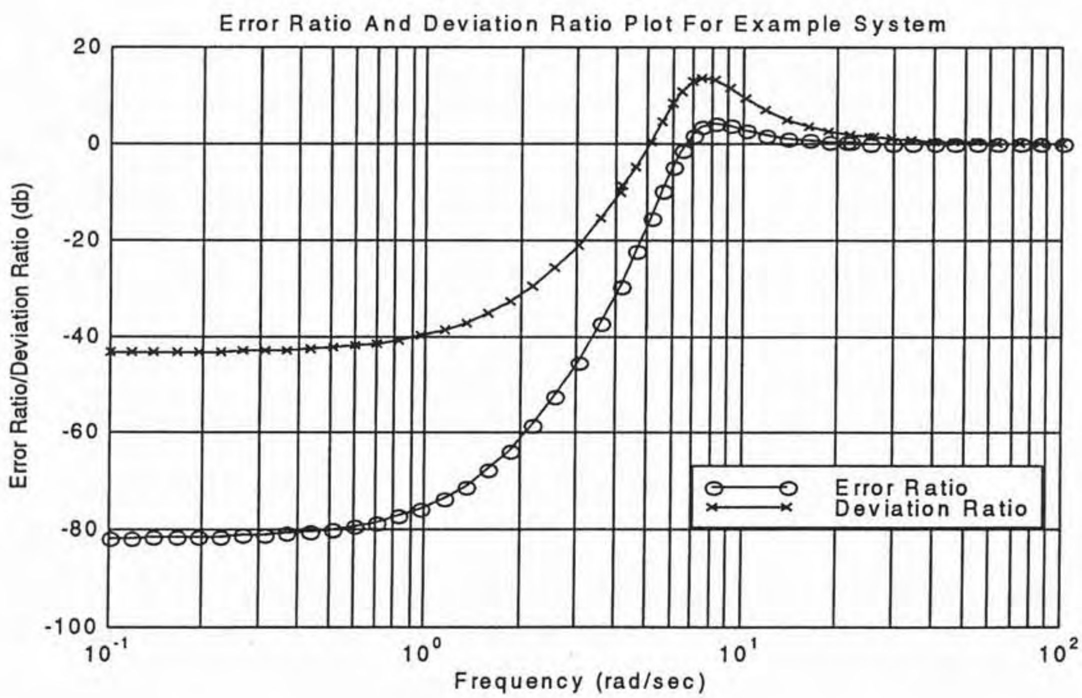
Max f is approx. 7 rad/sec from the graph.

Another method of finding the maximum stable frequency is the deviation ratio.

$$\text{Deviation Ratio} = \frac{\text{magnitude closed loop error}}{\text{magnitude setpoint}}$$

$$DR(s) = \left| \frac{1}{1 + G(s) \cdot H(s)} \right|$$

Plot Deviation and Error ratios on same graph for the example system.



Same zone criteria as error ratio

Stability of Control Systems From Bode Plots

A stable control system is one that has an open-loop gain of less than 1 (<0 db) when the open-loop phase angle is -180 degrees.

If control system has an open-loop gain of exactly 1 (0 db) when the open-loop phase angle is -180 degrees, then the system is marginally stable. (will oscillate).

If the control system has an open-loop gain of greater than 1 (>0 db) when the open-loop phase angle is -180 degrees, then the system is unstable.

Gain and Phase Margins

Gain Margin - Gain level that is considered a safe level below 1 (0 db). Minimum level set at 0.5 (-6db) at phase shift of -180 degrees.

Phase Margin - Phase shift above -180 degrees that is considered a safe level. Minimum level set at 40 degrees

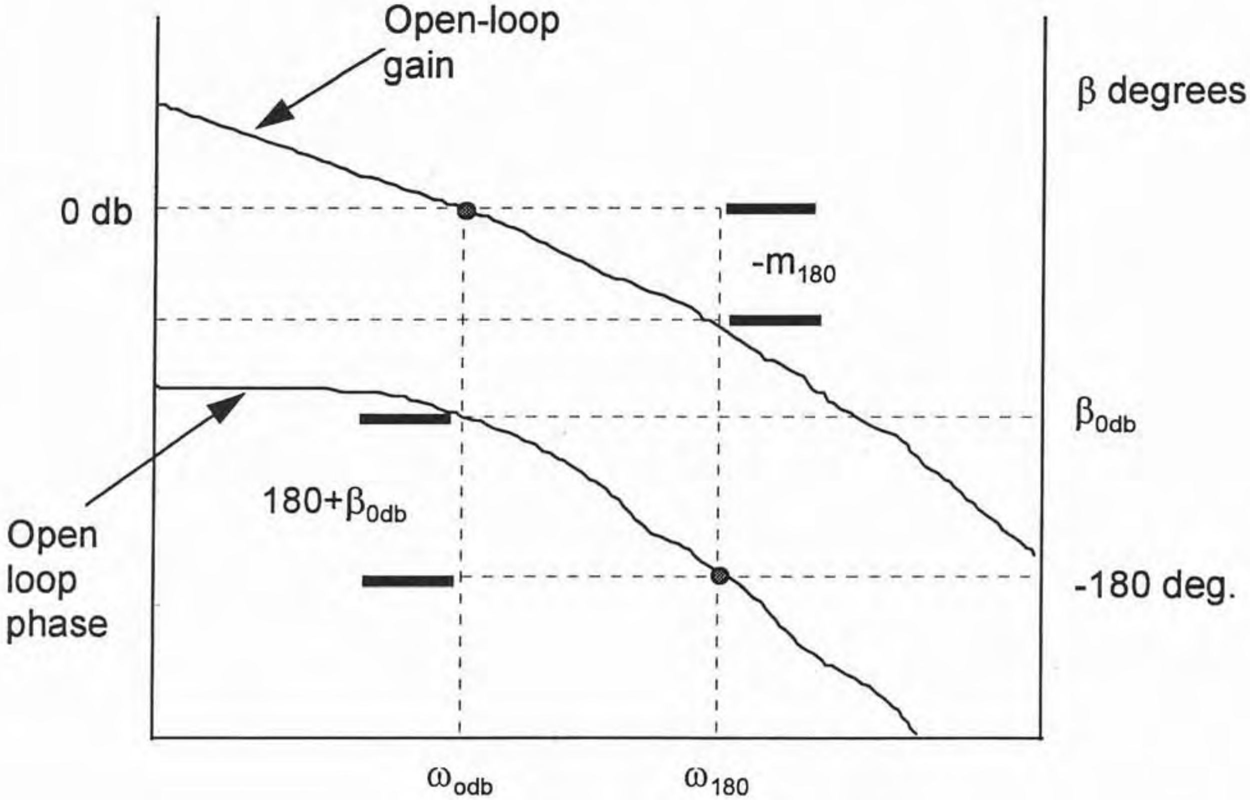
Define the two frequencies

ω_{0db} Frequency where gain is zero db

ω_{180} Frequency where the phase shift is -180 degrees

Gain and Phase Margins

$-m_{180}$ Gain margin $\qquad 180 + \beta_{0db}$ Phase margin



Procedure

1.) Draw vertical lines through the 0 db point of the gain plot and the -180 degree phase shift point of the phase plot

2.) Draw horziontal lines through the 0 db point of the gain plot and the -180 degree so that both ω_{0db} and ω_{180} lines are intersected.

3.) Read the gain and phase margins from the difference between the 4 horziontal lines

Example: A first order lag process has a dead-time delay of 2 seconds and is controlled by a proportional controller. The open-loop transfer function is given by:

$$GH(s) = 40 \cdot \frac{1}{(1 + 100 \cdot s)} \cdot e^{-2 \cdot s}$$

1.) Find the magnitude and phase angle of the transfer function at ω = 0.001, 0.01, 0.1 and 1 rad/sec by hand calculation.

Substitute $j\omega$ for s and find complex number. For the exponential term, remember Euler's Identity.

$$e^{-2j \cdot \omega} = \cos(2 \cdot \omega) - j \cdot \sin(2 \cdot \omega)$$

Gain always 1
phase increases
with increasing ω

For $\omega := 0.001$

$$GH(\omega) := 40 \cdot \frac{1}{(1 + 100j \cdot \omega)} \cdot e^{-2j \cdot \omega}$$

$$\frac{1}{1 + 100j \cdot 0.001} = 0.99 - 0.099j \qquad e^{-2j \cdot 0.001} = 1 - 2 \cdot 10^{-3} j$$

Convert to polar form $|0.99 - 0.099j| = 0.995$

$$\phi_l := \operatorname{atan}\left(-\frac{0.099}{.99}\right) \qquad \phi_l = -5.711 \cdot deg$$

Example: (cont.) Convert exponential part to magnitude and phase shift.

$$\left|1 - 2\cdot 10^{-3}\cdot j\right| = 1 \qquad \phi_d := \operatorname{atan}\left(\frac{2\cdot 10^{-3}}{1}\right) \qquad \phi_d = 0.115\cdot deg$$

Multiply magnitudes and add phase angles

$$m := 40\cdot 0.995\cdot 1 \qquad m = 39.8 \qquad \phi_t := \phi_l + \phi_d$$

Convert Gain to db $\quad db := 20\cdot \log(m)$

$db = 31.998$ db

$\phi_t = -5.596\cdot deg$ Phase shift

Now for $\omega := 0.01$

$$GH(\omega) := 40\cdot \frac{1}{(1 + 100j\cdot\omega)}\cdot e^{-2j\cdot\omega}$$

$$\frac{1}{1 + 100j\cdot 0.01} = 0.5 - 0.5j \qquad e^{-2j\cdot 0.01} = 1 - 0.02j$$

Convert to polar form $\quad |0.5 - 0.5j| = 0.707$

$$\phi_l := \operatorname{atan}\left(-\frac{0.099}{.99}\right) \qquad \phi_l = -5.711\cdot deg$$

$$\phi_l := \operatorname{atan}\left(-\frac{.5}{.5}\right) \qquad \phi_l = -45\cdot deg$$

Exponential gain and phase shift

$|1 - 0.02 \cdot j| = 1$ $\quad \phi_d := \operatorname{atan}\left(\frac{-0.02}{1}\right)$ $\quad \phi_d = -1.146 \cdot deg$

Multiply magnitudes and add phase angles

$m := 40 \cdot 0.707 \cdot 1$ $\quad m = 28.28$ $\quad \phi_t := \phi_l + \phi_d$

Convert Gain to db $\quad db := 20 \cdot \log(m)$

$db = 29.03$ db

$\phi_t = -46.146 \cdot deg$ Phase shift

Using the same method to find the values at ω=0.1 and 1 rad/sec gives the following results.

At $\omega := 0.1$

$db := 12.0$ db $\quad \phi_t := -95.7 \cdot deg$

At $\omega := 1.0$

$db := -8.0$ db $\quad \phi_t := -204.0 \cdot deg$

Construct an open-loop Bode plot using Matlab and find the gain and phase margins for the control system

Matlab code to find the Bode plot

```
% Example bode calculations
clear all;
close all;
% define the forward gain numerator and denominator coefficients
numgh=[40];
demgh=[100 1];
% construct the transfer function
Gh=tf(numgh,demgh);
% Use the bode function with its arguments so that it returns the
% magnitude, m, the phase shift, p and the frequencies so that
% the effect of the dead time delay can be added to the system
[m p w]=bode(Gh,{0.001,1});
% now compute the values of phase shift for the time delay using the
% formula -2*w*57.6
pd=-2*w*57.6;
% Add the the phase shift of the transfer function to the deadtime delay
% take the phase shift out of the 3 column array [d p w]
phase=p(:);
pt=pd+phase;
% compute the gain in db
db=20.*log10(m);
% create a figure window
figure;
% divide the plot area in two parts
subplot(2,1,1);
semilogx(w,db,'go-');
xlabel('Frequency (rad/sec)');
ylabel('Gain (db)');
title('Example Bode Plot');
grid on;
subplot(2,1,2);
semilogx(w,pt,'go-');
xlabel('Frequency (rad/sec)');
ylabel('Phase Shift (Degrees)');
grid on;
```