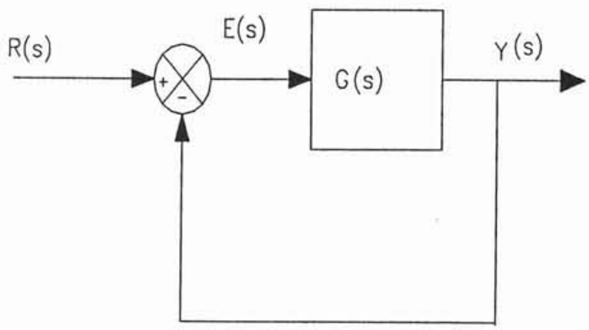
$$\frac{Y(s)}{R(s)} = G_1(s) + G_2(s)$$

Sum parallel connections of transfer functions

Systems with feedback - unity gain feedback



Write equations

$$Y(s) = G(s) \cdot E(s)$$

$$E(s) = R(s) - Y(s)$$

combine the above equations and simplify

$$Y(s) = G(s) \cdot (R(s) - Y(s))$$

$$Y(s) = G(s) \cdot R(s) - G(s) \cdot Y(s)$$

$$Y(s) + G(s) \cdot Y(s) = G(s) \cdot R(s)$$

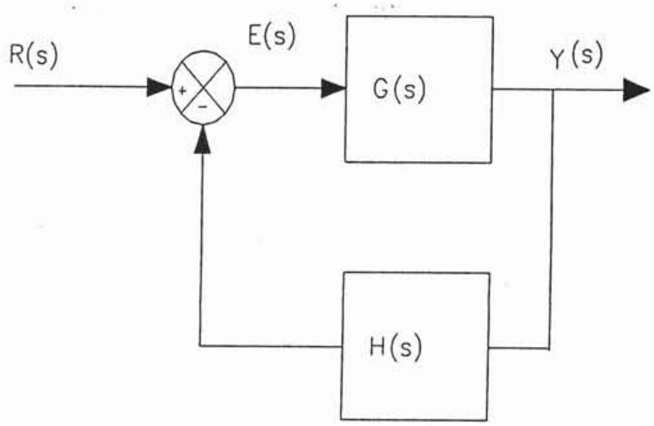
$$Y(s) \cdot (1 + G(s)) = G(s) \cdot R(s)$$

Place into transfer function form

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

Transfer function of system with unity gain feedback

For systems without non-unity gain feedback



$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s) \cdot H(s)}$$

G(s) = forward path gain
H(s) = feedback path gain

Bode Plots of Transfer Functions

Two Plots: a.) gain vs frequency semilog plot
Gain in db - linear scale. Frequency logarithmic
b.) phase shift vs frequency semilog plot
Phase shift - linear scale. Frequency logarithmic

Bode plot is frequency response of a control system

Gain $$G = \frac{output}{input}$$

For Bode Plot $$db = 20 \cdot \log(G)$$

Phase shift = output angle - input angle

To compute points for a Bode plot

1.) Replace s with $j\omega$ in transfer function.
2.) Select frequencies of interest (rad/sec) $\omega = 2\pi f$
3.) Compute magnitude and Phase angle of complex expression

Magnitude and phase of complex number

$$z = a + j \cdot b \qquad |z| = \sqrt{a^2 + b^2} \qquad \theta = \arctan\left(\frac{b}{a}\right)$$

Example: a self-regulating liquid tank has a transfer function of the form shown below. The tank has a time constant of τ = 1590 sec and a gain of 2000 s/m2. Determine the amplitude and phase shift of the system to a sinusoidal input that has a frequency of 0.0001592 Hz.

$$\frac{H(s)}{Q(s)} = \frac{G}{1 + \tau \cdot s}$$

Example: solution

Substitute in values into transfer function and replace s with jω

$G := 2000 \qquad \tau := 1590 \qquad f := 0.0001592 \cdot Hz$

$$\frac{H(j\cdot\omega)}{Q(j\cdot\omega)} = \frac{2000}{1 + 1590\cdot j\cdot\omega}$$

Convert frequency to rad/sec

$\omega := 2\cdot\pi\cdot f \qquad \omega := 2\cdot\pi\cdot 0.0001592 \qquad \omega = 0.001$

$$\frac{H(j\cdot\omega)}{Q(j\cdot\omega)} = \frac{2000}{1 + 1590\cdot j\cdot 0.001} \qquad \frac{H(j\cdot\omega)}{Q(j\cdot\omega)} = \frac{2000}{1 + 1.59\cdot j} \quad \text{rad/s}$$

$$\frac{H(j\cdot\omega)}{Q(j\cdot\omega)} = \frac{2000}{1 + 1.59\cdot j}$$ Simplify complex number to get

$$\frac{2000}{1 + 1.59j} = 566.877 - 901.335i$$ Convert to polar form

$a := 566.877 \qquad b := -901.335$

$mag\_z := \sqrt{a^2 + b^2} \qquad mag\_z = 1064.779$

$\theta := \operatorname{atan}\left(\frac{b}{a}\right) \qquad \theta = -57.833 \cdot deg$ The output lags input

$db := 20\cdot\log(1064.779) \qquad db = 60.545$

Gain of 60.545 at 0.001 rad/s

Using Matlab to Define Transfer Functions and Construct Bode Plots

To define Linear Time-invarient System (LTI)

Use the following Matlab functions from the command line

sys = zpk(z, p, k)   Turns arrays of coefficients into LTI
Linear time invarient system called sys
z = array of system zeros
p = array of system poles
k = array of system gains

bode(sys)   Plots the gain and phase plots of the system, sys

Given transfer function

$$\frac{V_o(s)}{V_i(s)} = \frac{0.005 \cdot s}{(0.001 \cdot s + 1) \cdot (0.001 \cdot s)}$$

Function has 1 zero at s = 0 and 2 poles s = -1/0.001 = 1000

Dividing top and bottom by 0.001 gives the transfer function in standard form

$$\frac{V_o(s)}{V_i(s)} = \frac{5 \cdot s}{(s + 1000) \cdot (s + 1000)} = \frac{5 \cdot s}{(s + 1000)^2}$$

Code to generate the bode plot. Type at the command line

```
k =[5]
p=[1000 1000]
z=[0]
sys=zpk(z,p,k)
bode(sys)
```

The bode(sys) function produces a plot window on the screen in Matlab that can be cut and pasted into wordprocessors and other programs.

To plot more than one transfer function on the same axis use the syntax: bode(sys1, sys2,..). The parameters are different system defined with the zpk(z,p,k) function

Change the gain of the original transfer function by a factor of 100 and plot both together on same graphs.

$$\frac{V_o(s)}{V_i(s)} = \frac{500 \cdot s}{(s+1000)\cdot(s+1000)} = \frac{500 \cdot s}{(s+1000)^2}$$

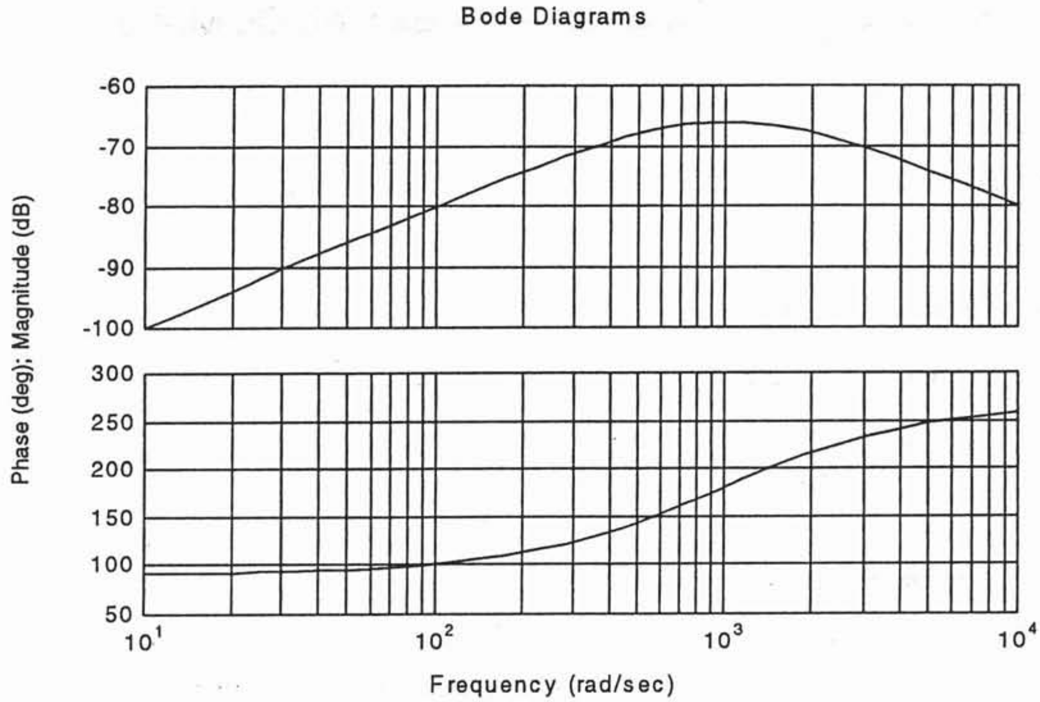
Code to generate bode plots

```
k=[5]
z=[0]
p=[1000 1000]
sys1=zpk(z,p,k)
k1=500
sys2=zpk(z,p,k1)
bode(sys1,sys2)
```

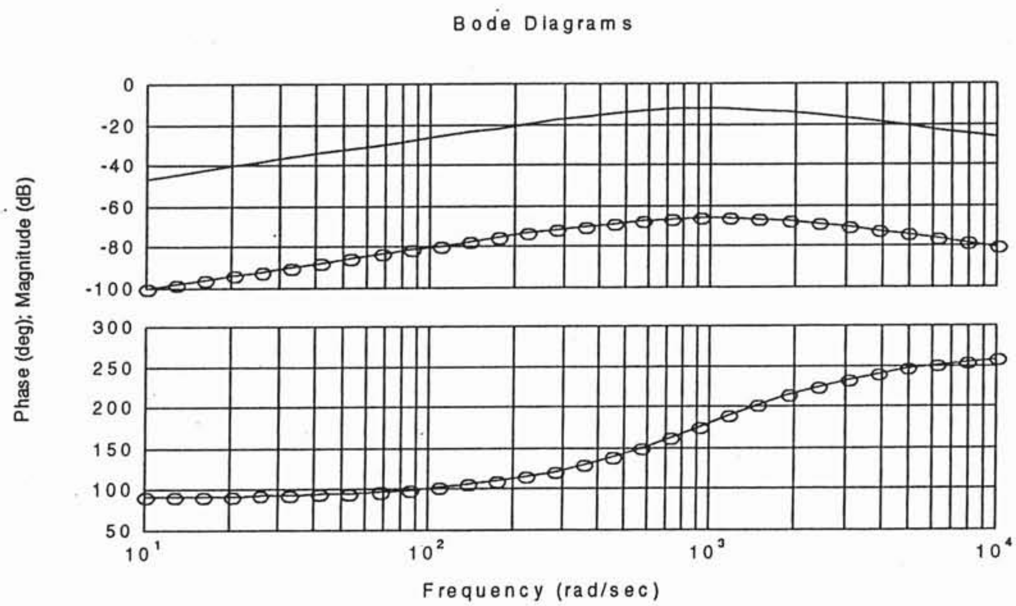
to change colors and line styles use

```
bode(sys1,'ro--',sys2,'b-')
```

The first style changes sys1 lines to red with circles at each data point. The line is dashed. The second style change the color of the line to blue and makes the line solid.



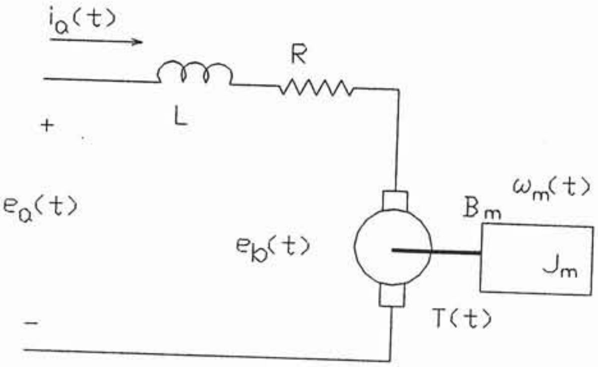
Plot with circles is sys1. Notice that gain change only affects the Magnitude of the response, not the phase shift.



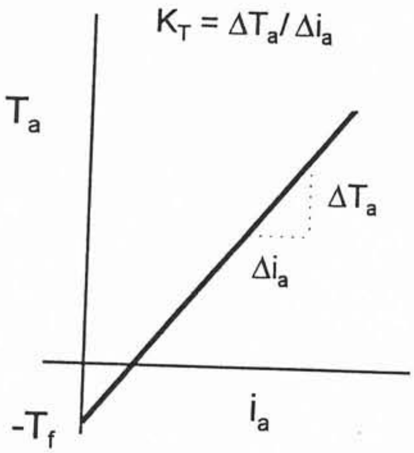
Steady-state Operation of a Separately Excited Dc Motor

Steady-state model

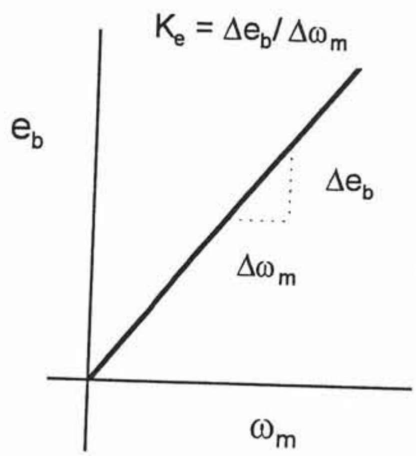
i_a = armature current
e_b = back emf
e_a = armature voltage
ω_m = motor speed rad/s
T = motor torque
T_f = static friction torque



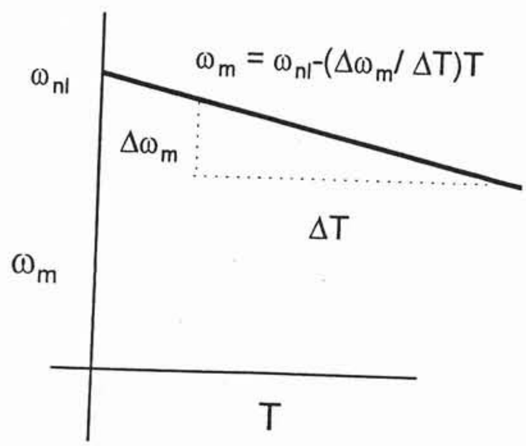
Current torque curve



Back emf curve



Speed torque curve



Steady-state Motor Equations

$$T = K_T \cdot i_a - T_f \quad \text{N-m} \qquad e_b = K_e \cdot \omega_m \quad \text{V}$$

$$e_a = i_a \cdot R + e_b \quad \text{V} \qquad P = \omega_m \cdot T \quad \text{W}$$

Combining the above relationships gives

$$\frac{K_T \cdot e_a - (T - T_f) \cdot R}{K_T \cdot K_e} = \omega_m \qquad \frac{e_a - i_a \cdot R}{K_e} = \omega_m$$

If load torque is zero then the above equation gives the no-load speed.

$$\frac{K_T \cdot e_a - (T_f) \cdot R}{K_T \cdot K_e} = \omega_{nl}$$

Example: Armature controlled dc motor has the following ratings T_f = 0.012 N-m, K_T = 0.06 N-m/A K_e = 0.06 V-s/rad R = 1.2 ohms maximum speed of 500 rad/sec with a maximum current of 2 amps find: a.) maximum output torque, b.) maximum output power c.) maximum armature voltage, d. no-load speed at maximum armature voltage

a.) T_{max} occurs at I_{max} $\quad T_{max} = K_T I_{max} - T_f = (0.06\ \text{N-m/A})2A - 0.012\ \text{N-m}$

$T_{max} = \boxed{0.108\ \text{N-m}}$ ANS

b.) $P_{max} = \omega_{max} T_{max} = (500\ \text{Rad/s})(0.108\ \text{N-m}) = \boxed{54\ W}$ ANS

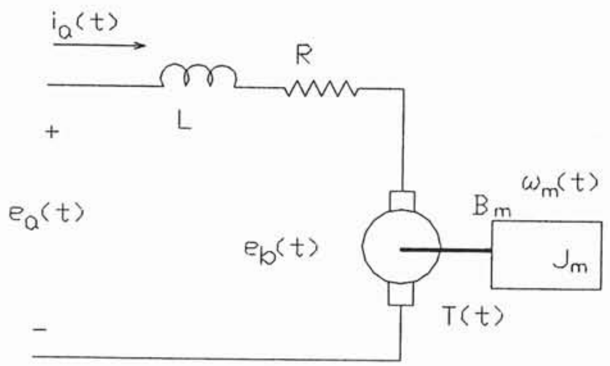
c.) e_a maximum $\quad e_a = i_a R + e_b \Rightarrow e_b = i_a R + K_e \omega_{max}$

$e_a = 1.2\Omega(2A) + (0.06\ \text{V-s/rad})(500\ \text{rad/s}) = \boxed{32.4V}$ ANS

d.) $\frac{K_T e_a - (T + T_f)R}{K_e K_T} = \omega_m$ at no Load $T = 0$ $\quad \frac{K_T e_a - T_f R}{K_e K_T} = \omega_{NL}$

$$\frac{(0.06)(32.4) - (0.012)(1.2)}{(0.06)(0.06)} = \boxed{536\ \text{rad/s} = \omega_{NL}}$$

Transfer function model of a armature controlled Dc motor



Write electrical equations and mechanical equation use electromechanical relationships to couple the two sets of equations

E_a and E_b are inputs I_a is output

$e_a(t) = L \cdot \frac{d}{dt} i_a(t) + R \cdot i_a(t) + e_b(t)$ KVL around armature loop

$T(t) = J_m \cdot \frac{d}{dt} \omega_m(t) + B_m \cdot \omega_m(t)$ Mechanical dynamics

$e_b(t) = K_e \cdot \omega_m(t) \qquad T(t) = K_T \cdot i_a(t)$

Find tranfer function between armature voltage and motor speed $\frac{\Omega(s)}{E_a(s)}$

Take Laplace of above equations and write in I/O form

$$E_a(s) = L \cdot s \cdot I_a(s) + R \cdot I_a(s) + E_b(s)$$

$$I_a(s) = \frac{1}{L \cdot s + R} \cdot \left(E_a(s) - E_b(s)\right)$$

$T(s) = K_T \cdot I_a(s) \qquad E_b(s) = K_e \cdot \Omega(s)$

$T_a(s) = J_m \cdot s \cdot \Omega(s) + B_m \cdot \Omega(s)$ Laplace of mechanical equation

Rewrite mechancial equation as an I/O equation

$$T(s) = \left(J_m \cdot s + B_m\right) \cdot \Omega(s)$$

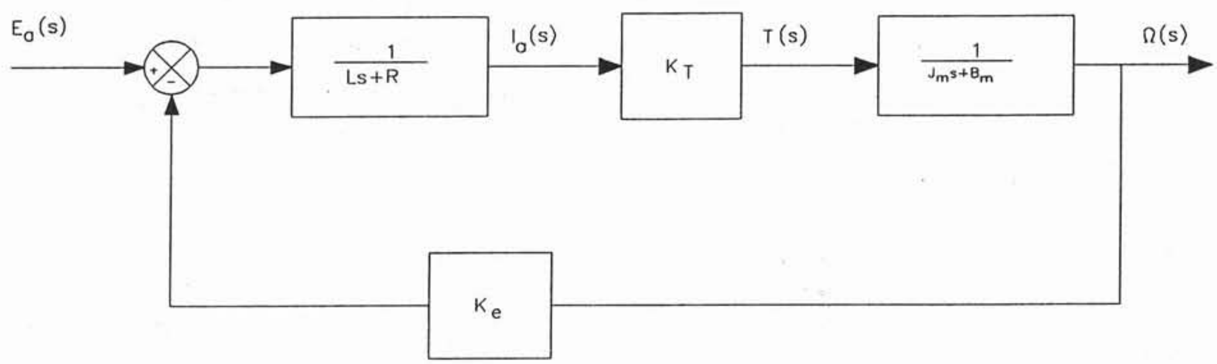
Draw block diagrams from these equations

$$\Omega(s) = \frac{1}{\left(J_m \cdot s + B_m\right)} \cdot T_a(s)$$

$$E_b(s) = K_e \cdot \Omega(s)$$

$$I_a(s) = \frac{1}{L \cdot s + R} \cdot \left(E_a(s) - E_b(s)\right)$$

$$T(s) = K_T \cdot I_a(s)$$



Note that the dc motor has an inherent feedback from the cemf this can improve the stability of the system be adding an electro mechancial damping

To get transfer function must reduce the system using the feedback formula. $H(s) = K_e$

G(s) is the product of all the blocks in the forward path

$$\frac{\Omega(s)}{E_a(s)} = \frac{G(s)}{1 + G(s) \cdot H(s)} \qquad G(s) = \frac{K_T}{(L \cdot s + R)} \cdot \frac{1}{(J_m \cdot s + B_m)}$$

$$H(s) = K_e \qquad G(s) = \frac{K_T}{(L \cdot s + R) \cdot (J_m \cdot s + B_m)}$$

$$\frac{\Omega(s)}{E_a(s)} = \frac{\dfrac{K_T}{(L \cdot s + R) \cdot (J_m \cdot s + B_m)}}{1 + \dfrac{K_T}{(L \cdot s + R) \cdot (J_m \cdot s + B_m)} \cdot K_e}$$

Simplify by multiplying top and bottom by the factors $(Ls+R)(J_m s+B_m)$

$$\frac{\Omega(s)}{E_a(s)} = \frac{K_T}{(L \cdot s + R) \cdot (J_m \cdot s + B_m) + K_T \cdot K_e}$$

Expand factors and collect like terms of s

$$\frac{\Omega(s)}{E_a(s)} = \frac{K_T}{L \cdot J \cdot s^2 + (B_m \cdot L + R \cdot J_m) \cdot s + (K_T \cdot K_e + R \cdot B_m)}$$

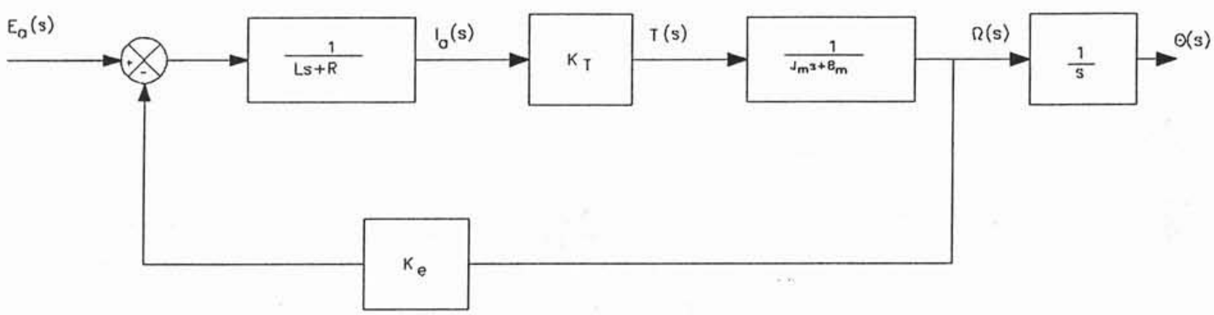
Roots of the denominator can be effected by values or R, B_m, K_T J_m and K_e Second order system roots could have imaginary component.

In SI units K_T numerically equal to K_e

To find the motor position with respect to the armature voltage, the following block diagram must be reduced.

To find position, integrate the velocity

$$\frac{d}{dt}\theta = \omega \qquad \int \left(\frac{d}{dt}\theta\right) dt = \int \omega \, dt = \theta$$



$$\Theta(s) = \frac{1}{s} \cdot \Omega(s)$$ Positon found by multiplication by 1/s

Define $\frac{J_m}{B_m} = \tau_m$ mechanical time constant

$\frac{L}{R} = \tau_e$ electrical time constant

Electrical time constant much small than mechanical time constant. Usually set to zero. Transfer function reduces to

$$\frac{\Omega(s)}{E_a(s)} = \frac{K_s}{1 + \tau_s \cdot s} \qquad K_s = \frac{K_T}{R \cdot B_m + K_e \cdot K_T} \qquad \tau_s = \frac{R \cdot J_m}{R \cdot B_m + K_T \cdot K_e}$$

Consider motor and load - Load has inertia J_L and viscous friction B_L. Consider load connect through a speed reducer motor connected to smaller gear with N_1 teeth. load to larger gear with N_2 teeth.

Gear ratio similar to transformer action

$\omega_L = \frac{N_1}{N_2} \cdot \omega_m$ rad/sec $\qquad T_L = \frac{N_2}{N_1} \cdot T_m$ N-m

Effects on friction and inertia

Without speed change $\qquad B_T = B_m + B_L$ N-m-s/rad

$J_T = J_m + J_L$ N-m-s^2/rad

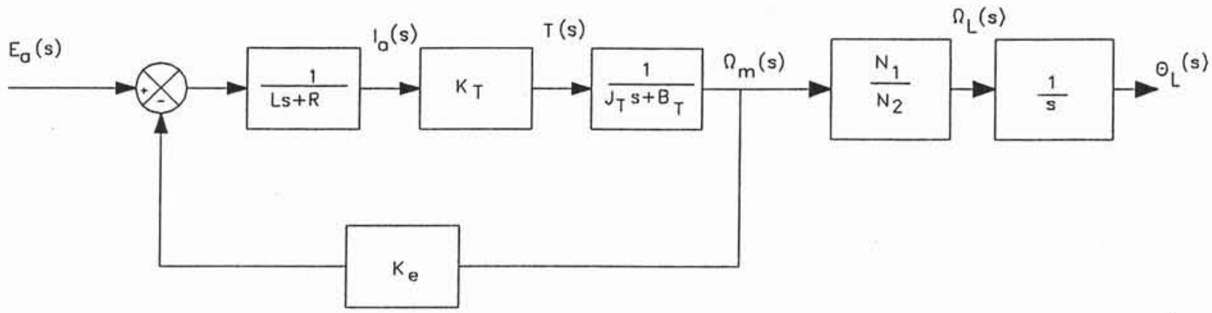
With speed change

$$B_T = B_m + \left(\frac{N_1}{N_2}\right)^2 \cdot B_L \quad \text{N-m-s/rad}$$

$$J_T = J_m + \left(\frac{N_1}{N_2}\right)^2 \cdot J_L \quad \text{N-m-s}^2\text{/rad}$$

Speed changer reduces speed but increases torque. Increases affects of both load friction and inertia

New block diagram



$$\frac{\Omega_L(s)}{E_a(s)} = \frac{K_T \cdot \left(\frac{N_1}{N_2}\right)}{L \cdot J_T \cdot s^2 + \left(B_T \cdot L + R \cdot J_T\right) \cdot s + \left(K_T \cdot K_e + R \cdot B_T\right)}$$

Example: A dc permanent magnet motor has the following specifications: Maximum speed = 500 rad/s
Maximum armature current = 2.0 A
Voltage constant = 0.06 V-s/rad
Torque constant = 0.06 N-m/A
Friction torque = 0.012 N-m
Armature resistance = 1.2 ohm
Armature inductance = 0.02 H
Armature inertia = 6.2×10^{-4} N-m-s^2/rad
Armature viscous friction = 1×10^{-4} N-m-s/rad

a.) Determine the voltage/velocity and voltage/position transfer functions for this motor
b.) Determine the voltage/velocity and voltage/position transfer functions for the motor neglecting the electrical time constant

Example Solution

Full model of motor $\frac{\Omega_m(s)}{E_a(s)} = \frac{K_T}{(RB_m + K_E K_T) + (RJ_m + B_m L)s \quad LJ_m s^2}$

$K_T = 0.06 \quad K_E = 0.06 \quad J_m = 6.2\times10^{-4}$

$R = 1.2\Omega \quad L = 0.02H \quad B_m = 1\times10^{-4}$

$RB_m + K_E K_T = 1.2(1\times10^{-4}) + 0.06(0.06) = 0.00372$

$RJ_m + B_m L = 1.2(6.2\times10^{-4}) + (1\times10^{-4})(0.02) = 7.46\times10^{-4}$

$LJ_m = 0.02(6.2\times10^{-4}) = 1.24\times10^{-5}$

$$\frac{\Omega_m(s)}{E_a(s)} = \frac{0.06}{0.00372 + 7.46\times10^{-4}s + 1.24\times10^{-5}s^2}$$

divide Top and bottom by 0.00372

$$\frac{\Omega_m(s)}{E_a(s)} = \boxed{\frac{16.13}{1 + 0.201s + 0.00333s^2}}$$

To convert to Position integrate velocity (divide by s)

$\Theta_m(s) = \frac{1}{s}\Omega_m(s)$

$$\Theta_m(s) = \frac{1}{s}\left[\frac{16.13}{1 + 0.201s + 0.00333s^2}\right] = \boxed{\frac{16.13}{s + 0.201s^2 + 0.00333s^3}}$$

b.) Ignoring Electrical Time constant $\frac{\Omega_m(s)}{E_a(s)} = \frac{K_s}{1 + \tau_s s}$

$$K_s = \frac{K_T}{RB_m + K_E K_T} = \frac{0.06}{1.2(1\times10^{-4}) + (0.06)(0.06)} = 16.13$$

$$\tau_s = \frac{RJ_m}{RB_m + K_e K_T} = \frac{1.2(6.2\times10^{-4})}{1.2(1\times10^{-4}) + (0.06)(0.06)} = 0.2$$

$$\frac{\Omega_m(s)}{E_a(s)} = \boxed{\frac{16.13}{1 + 0.2s}}$$

$\Theta_m(s) = \frac{1}{s}\Omega_m(s)$

$$\frac{\Theta_m(s)}{E_a(s)} = \boxed{\frac{16.13}{s + 0.2s^2}}$$

Modes of Control

Proportional Control

Process characteristics for optimum Results

1.) Small Capacitance
2.) Rapid load Changes

Need gain high enough to reduce steady-state error to acceptable level.

Defining Equations

Time function $$v = K_p \cdot e + v_o$$

Laplace function $$V(s) = K_p \cdot E(s)$$

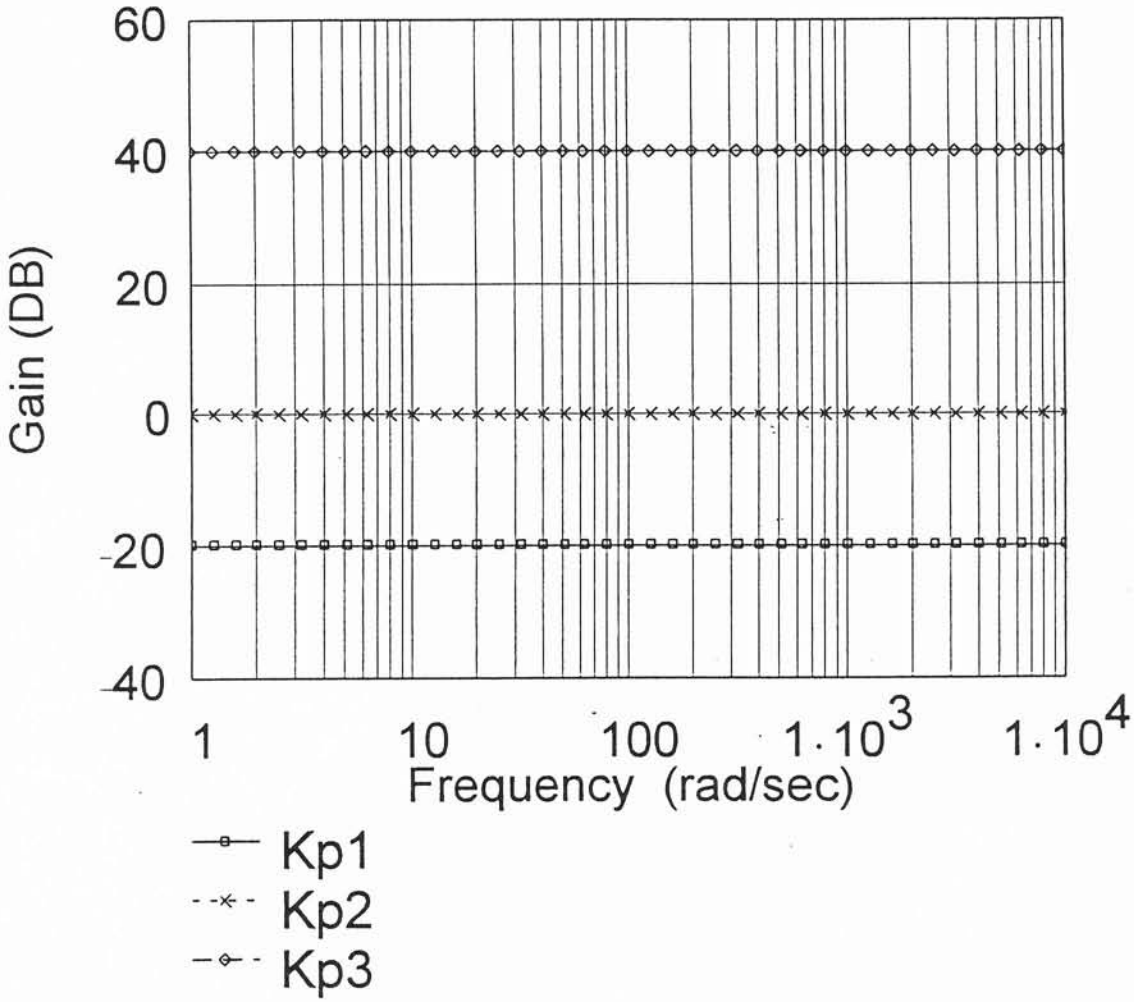
Transfer function $$\frac{V(s)}{E(s)} = K_p$$

Initial condition $v_o = 0$ on Laplace functions

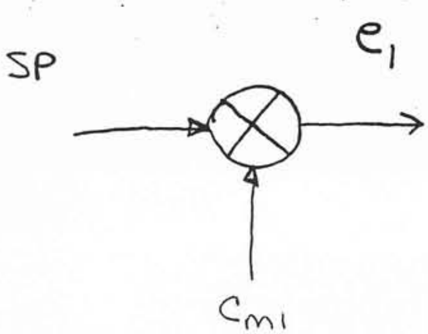
Where
- e = time domain error signal
- K_p = proportional gain
- v_o = controller output with $e=0$
- v = controller output

Bode plots of proportional controller

$K_{p1} := .1$ $K_{p2} := 1$ $K_{p3} := 100$



Determine how feedback affects performance

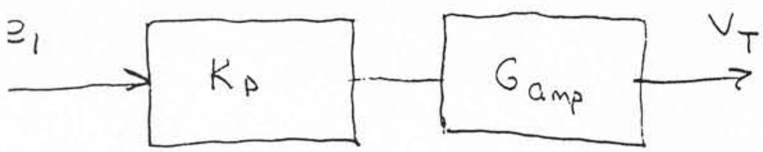


Find error and setpoint

$C_{m1} = K_{tac}\omega_1 = 0.11\ V/\text{rad-sec}\,(300\ \text{rad/sec})$

$C_{m1} = 33\ V$ Tach output

$e_1 = (SP - C_{m1})$ $V_T = 19.24\ V$ Power supply output



$$\frac{V_T}{K_p G_{amp}} = e_1$$

$$\frac{V_T}{K_p G_{amp}} = (SP - C_{m1}) \Rightarrow \frac{V_T}{K_p G_{amp}} + C_{m1} = SP$$

Plug in values

$K_p G_{amp} = 10\ V/V$

$$\frac{19.24V}{10\ V/V} + 33V = \boxed{34.924V = SP}$$

Error voltage $e_1 = (SP - C_{m1})$

$$e_1 = (34.924 - 33) = \boxed{1.924\ V}$$

Setpoint of 34.924 produces speed of 300 rad/sec with $T_L = 0.05$ N-m

when torque increases new V_T defined by

$$V_{T2} = (SP - C_{m1})K_p G_{amp} \Rightarrow V_{T2} = (SP - K_{tac}\omega_2)\,K_p G_{amp}$$

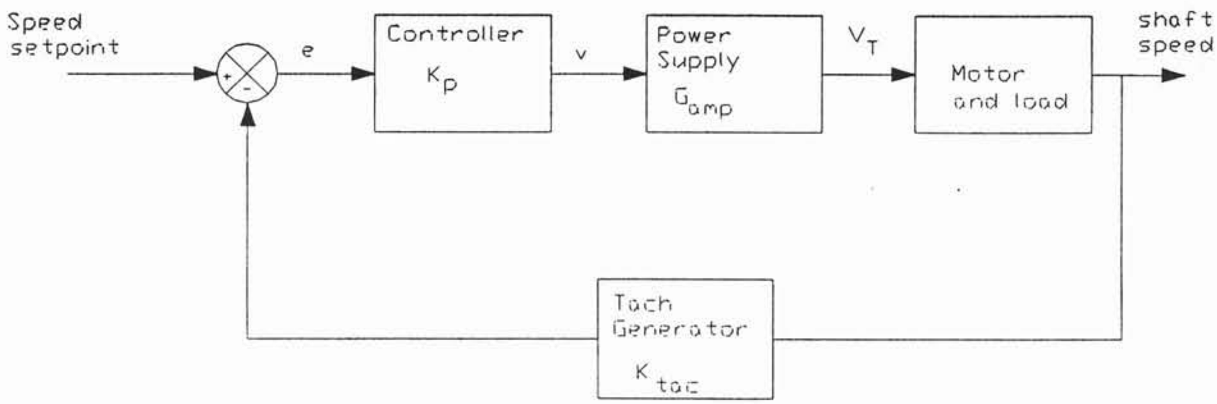
$SP_1 = SP_2$ error must change to increase V_T

For motor $e_b = K_e \omega_2$ back emf

$V_{T2} = I_{a2}R + e_{b2}$ substitue above

$$V_{T2} = I_{a2}R + K_e\omega_2$$

Example: Motor Speed Control



With open loop control torque change produces following results

$T_{L1} = 0.05$ N-m

$V_T = 19.24$

$I_{a1} = 1.033$ A

$\omega_1 = 300$ rad/sec

$T_{L2} = 0.075$ N-m

$V_T = 19.24$

$I_{a2} = 1.45$ A

$\omega_2 = 291.7$ rad/sec

Motor parameters

$T_f = 0.012$ N-m

$K_T = 0.06$ N-m/A

$R = 1.2$ Ohms

$K_e = 0.06$ V/rad-sec

$K_{tac} = 0.11$ V/rad-sec

$K_p \cdot G_{amp} = 10$ V/V

Example: (cont.)

$$V_{T2} = (SP - K_{tac}\omega_2) K_p G_{amp}$$

$$V_{T2} = (34.1924 - 0.11\omega_2)\, 10$$

$$V_{T2} = 341.924 - 1.1\omega_2 \quad (1)$$

Also with $T_L = 0.075$ N-m $I_{a2} = 1.45$ A

$$V_{T2} = (1.45\,A)(1.2\,\Omega) + 0.06\,\omega_2$$

$$V_{T2} = 1.74 + 0.06\,\omega_2 \quad (2)$$

Solve (1) and (2) simultaneously for ω_2 and V_{T2}

$$V_{T2} + 1.1\omega_2 = 349.24 \quad (1)$$ standard form

$$V_{T2} - 0.06\omega_2 = 1.74 \quad (2)$$

$V_{T2} = 19.71$ V $\omega_2 = 299.6$ rad/s

determine error signal $SP_1 = SP_2 = 34.924$ V

$$C_{m2} = K_{tac}\omega_2 = (0.11\,V/\text{rad-s})(299.6\,\text{rad/s}) = 32.956$$

$$e_2 = (SP_2 - C_{m2}) = (34.924 - 32.956) = 1.968$$ error voltage increases. Check speed error.

Setpoint 300 rad/s

Open loop Load Change

$$\frac{\omega_1 - \omega_2}{\omega_1} \times 100\%$$

$$\frac{(300 - 291.7)\,\text{rad/s}}{300\,\text{rad/s}} \times 100\% = 2.767\%$$

With feedback

$$\frac{(300 - 299.6)\,\text{rad/s}}{300\,\text{rad/s}} \times 100\% = 0.143\%$$

feedback reduces speed error by factor of 19.35

Integral Control Mode

Produces an output that is proportional to the integral of the error input. Adds a pole to system at s=0. Pure integrators tend to make system less stable.

Equations

time $$v(t) = K_I \cdot \int_0^t e(t)\,dt + v_o$$

Laplace $$V(s) = \frac{K_I}{s} \cdot E(s)$$

Transfer Function $$\frac{V(s)}{E(s)} = \frac{K_I}{s}$$

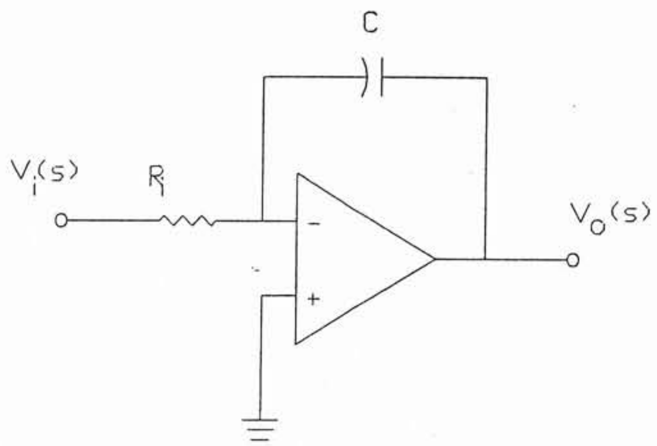
Note: initial conditions zero for transfer function model as per theory.

Where:

K_I = integral gain (proportionality constant)

OP AMP Realization of Integral Control

Ideal OP AMP Intergrator



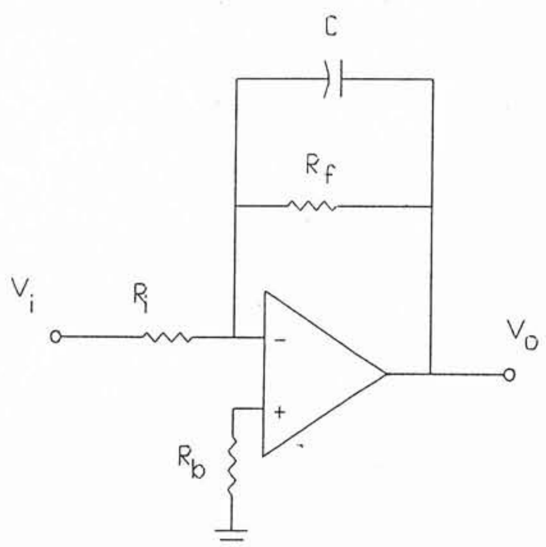
Transfer Function

$$V_o(s) = \frac{-1}{R_i \cdot C \cdot s} \cdot V_i(s)$$

One pole at

$$s = 0$$

Practical OP AMP integrators



$$A_V(s) = -\frac{-R_f}{R_i \cdot \left(1 + R_f \cdot C \cdot s\right)}$$

One pole at

$$s = -\frac{1}{R_f \cdot C}$$