To find frequency response, substitute jω for s and find the magnitude and phase shift of the transfer function for different values of f.

$$A_v(\omega) = \frac{-1}{R \cdot C \cdot j \cdot \omega}$$

Take magnitude of this function and phase angle

z = complex value

re(z) = real part of z im(z) = imaginary part of z

Magnitude

Phase Shift

Scale gain for db

$$|z| = \sqrt{re(z)^2 + im(z)^2} \qquad \phi = \arctan\left(\frac{im(z)}{re(z)}\right)$$

$$db(\omega) = 20 \cdot \log(|z(\omega)|)$$

Gain and Phase Plot of Practical OP AMP Integrator

$$A_v(\omega) = \frac{-R_f}{R_i \cdot (1 + R_f \cdot C \cdot j \cdot \omega)}$$

Practical integrator only exhibits integral action for frequencies much higher (10 times) $1/R_fC$.

$$\left|A_v(\omega)\right| = \frac{R_f}{R_i} \cdot \frac{1}{\sqrt{1 + R_f^2 \cdot C^2 \cdot \omega^2}}$$ Gain

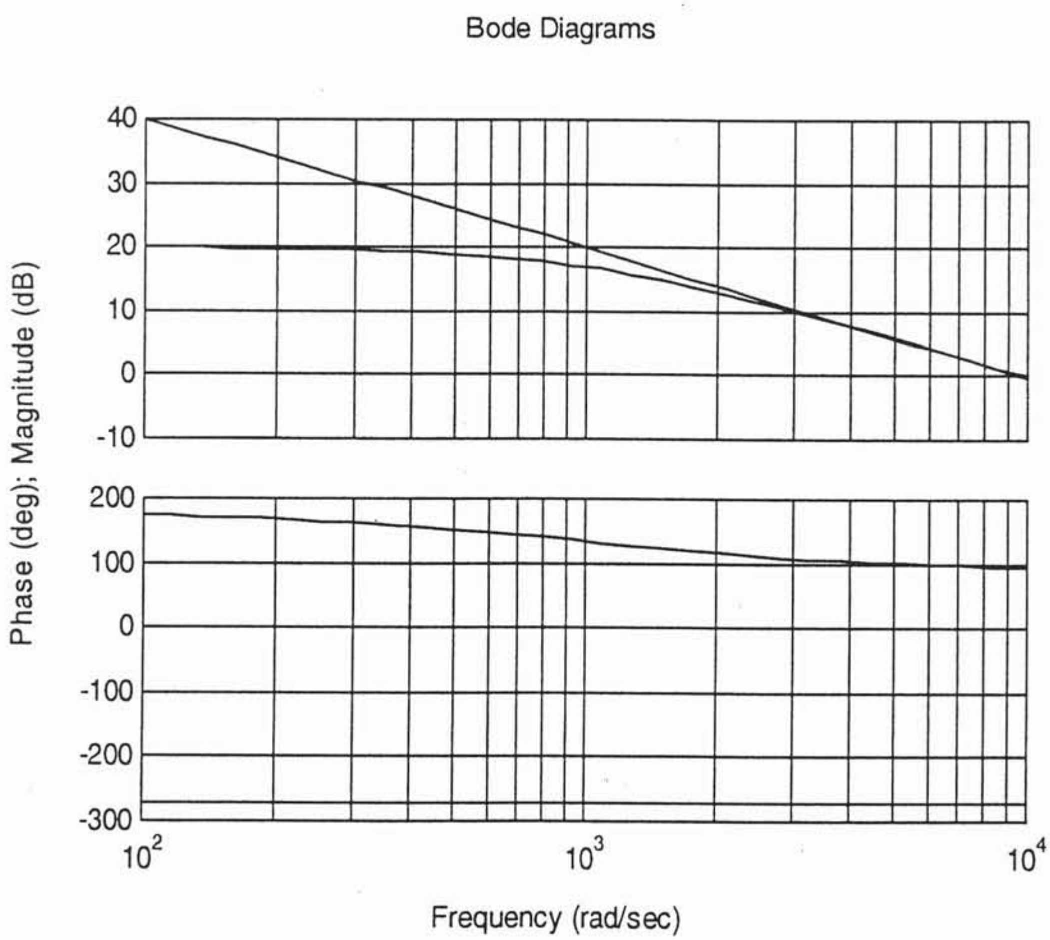
$$\phi(\omega) = 180 - \operatorname{atan}\left(R_f \cdot C \cdot \omega\right) \cdot \frac{180}{\pi}$$ Phase Shift

atan = arctan = $\tan^{-1}$

Using Matlab script to generate Bode plots and transfer function.

```
ri=input('Enter value of input resistance: ');
c=input('Enter value of capacitance: ');
rf=input('Enter value of feedback resistance: ');

% compute transfer function model parmeters for
paractical
% integrator

tau=rf*c;
ki=-rf./ri;
```

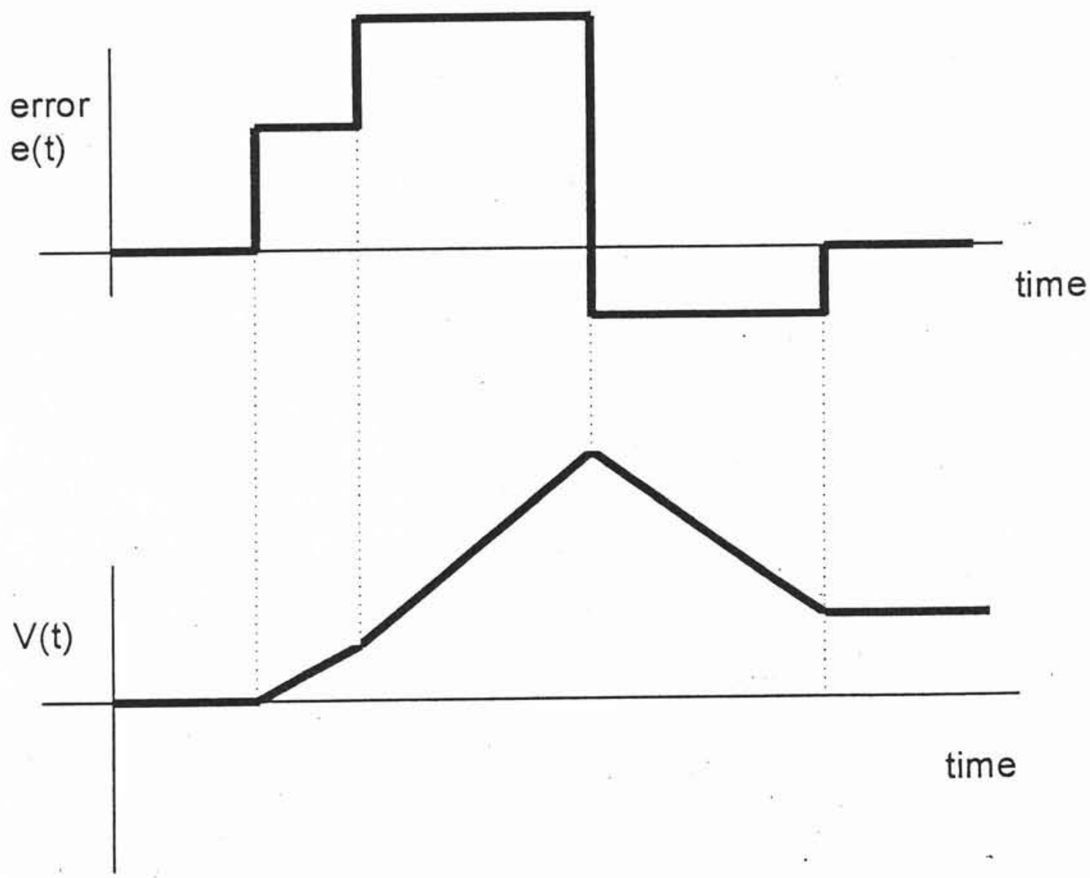
```
% compute parameter for ideal integrator
tau1 = ri*c;

% build transfer function
% denominator form a1*s^2+a2s+a3
Av=tf([ki],[tau 1])
Av1=tf([-1],[tau1 0])
%plot both on the same graphs
bode(Av,Av1);
```



Previous plots made for values

$R_i = 10k$ $R_f = 100k$ $C = 0.01\ \mu F$

Integtral action on time varying error signals



Integral of constant is line. Integrator produces linearly increasing output for constant error input. Negative error causes output to decrease. Zero error maintains value.

Example: an ideal integrator has a gain of $K_I = 0.1$ V/s. Its inital output at t=0 is v=1.5 V. determine the outputs if the error has step inceases of

e(t) = 0	0<=t<=1 s
e(t) = 2.5	1<t<=2 s
e(t) = 4	2<t<=3 s
e(t) = 0	3<t<=4 s
e(t) = -1.5	4<t<=5 s

$V_i(t) = K_I \int_{t_i}^{t} e(t)\,dx + V_0$ for 1st segment $t_1 = 0$

$V_1(t) = 0.1 \int_0^t 0\,dx + 1.5 = 0 + 1.5$ $\boxed{V_1(t) = 1.5}$ $0 \le t < 1$

$V_2(t) = 0.1 \int_1^t 2.5\,dx + 1.5 = 0.25x \Big|_1^t + 1.5 = 0.25t - 0.25 + 1.5$

$\boxed{V_2(t) = 0.25t + 1.25}$ $1 \le t < 2$ find inital value next step

$V_2(2) = 0.25(2) + 1.25 = 1.75$ V(0) for next stage

$V_3(t) = 0.1 \int_2^t 4\,dx + 1.75 = 0.4x \Big|_2^t + 1.75 = 0.4t - 0.8 + 1.75$

$\boxed{V_3(t) = 0.4t + 0.95}$ $2 \le t < 3$ Value at $t = 3$ $V_3(3) = 0.4(3) + 0.95$

$V_3(3) = 2.15$ initial condition for next step

$V_4(t) = 0.1 \int_3^t 0\,dx + 2.15 = 2.15$ $3 \le t < 4$ no change in error

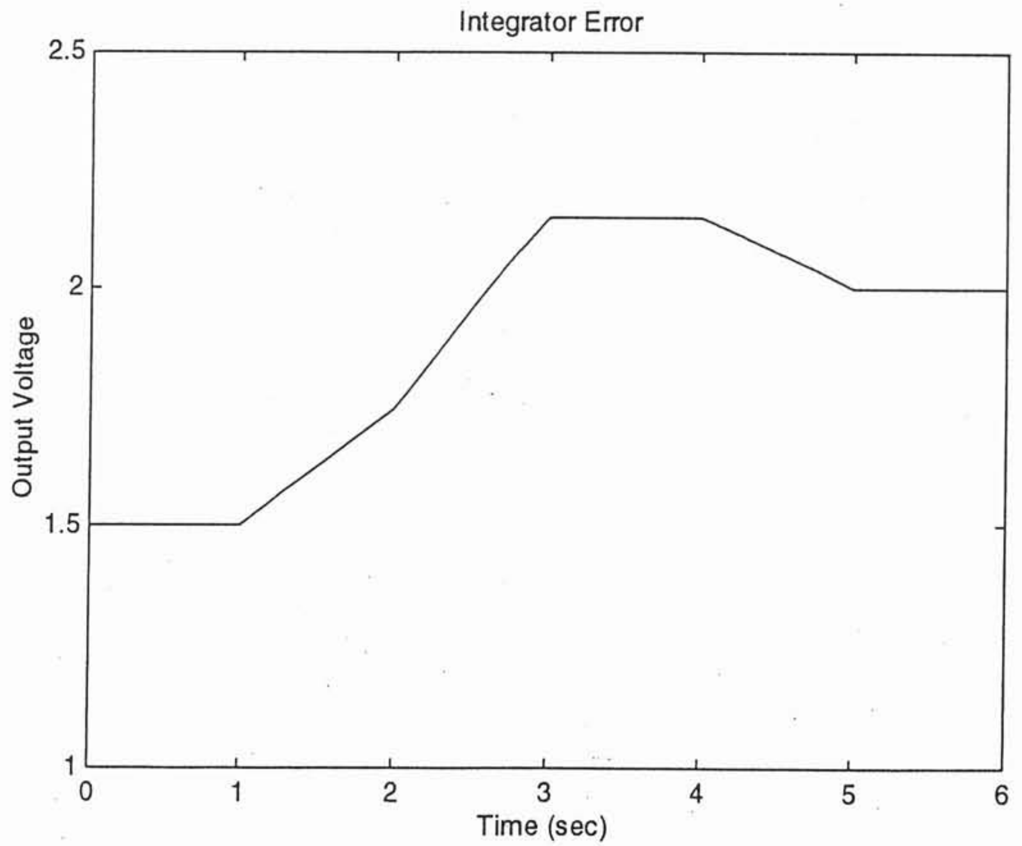
$V_5(t) = 0.1 \int_4^t -1.5\,dx + 2.15 = -0.15x \Big|_4^t + 2.15 = -0.15t + 0.6 + 2.15$

$\boxed{V_5(t) = -0.15t + 2.75}$ at $t = 5$ $V_5(5) = -0.15(5) + 2.75 = \underline{\underline{2}}$

OUTPUT DECREASES

Plot of integrator outputs   Matlab Code

```
t1=linspace(0,1,5)
x=ones(1,length(t1))
x1=x.*1.5
t2=linspace(1,2,5);
x2=0.25.*t2+1.25;
t3=linspace(2,3,5);
x3=0.4.*t3+0.95;
t4=linspace(3,4,5);
x4=ones(1,length(t4));
x4=x.*2.15;
t5=linspace(4,5,5);
x5=-0.15.*t5+2.75;
t6=linspace(5,6,5);
x6=x.*2;

plot(t1,x1,t2,x2,t3,x3,t4,x4,t5,x5,t6,x6);
axis([0 6 1 2.5]);
title('Integrator Error');
xlabel('Time (sec)');
ylabel('Output Voltage');
```



Proportional-Integral Control (PI)

Time Function

$$v(t) = K_p \cdot e(t) + K_p \cdot K_I \cdot \int_0^t e(t)\,dt + v_o$$

Laplace Function

$$V(s) = K_p \cdot E(s) + \frac{K_p \cdot K_I}{s} \cdot E(s)$$

Transfer Function

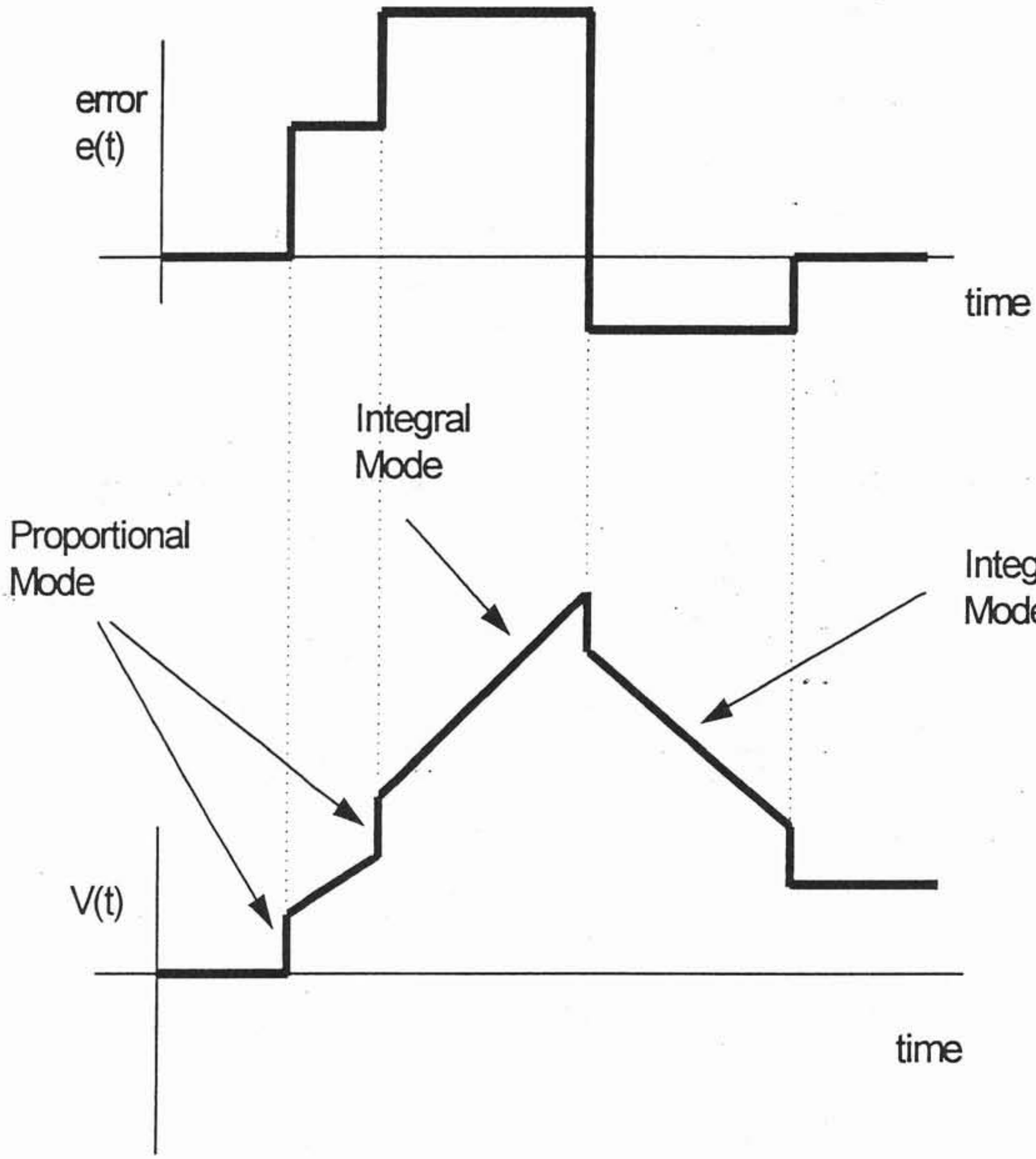
$$\frac{V(s)}{E(s)} = K_p \left(\frac{K_I + s}{s} \right)$$

Adds one pole and one zero to the transfer function of the system.

PI control used on process with large load changes Proportional action only can not reduce error to reasonable limits.

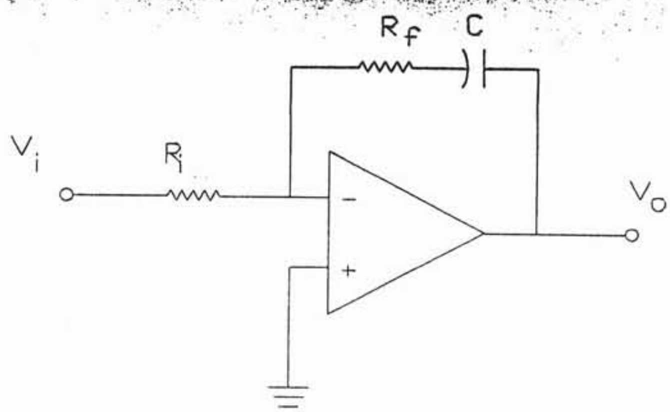
Intergral mode provides reset action that drives steady-state error to zero. Eliminates offset error

Time plots of PI control output



Proportional mode gives instantaneous response to error while integral mode decreases error over time

OP AMP Realization of PI Controller



Transfer Function $$A_v(s) = \frac{-Z_f(s)}{Z_i(s)} = -\frac{R_f + \frac{1}{C\cdot s}}{R_i}$$

Simplify $$A_v(s) = -\frac{R_f\cdot C\cdot s + 1}{R_i\cdot C\cdot s} = -\left(\frac{R_f}{R_i} + \frac{1}{R_i\cdot C\cdot s}\right)$$

Proportional Gain $$K_p = \frac{R_f}{R_i} \qquad K_I = \frac{1}{R_i\cdot C}$$

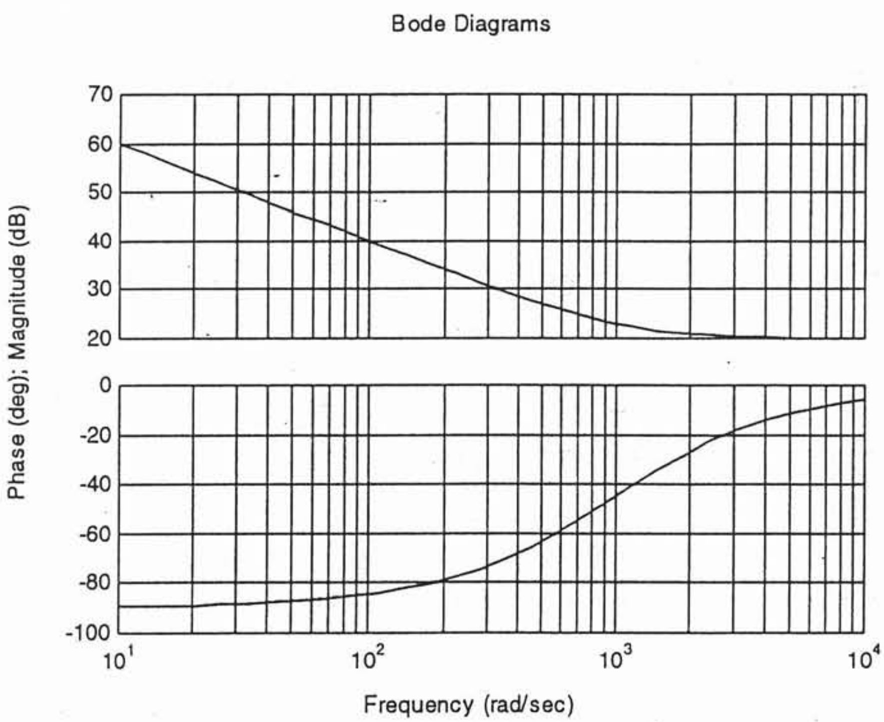
Select Values of R_i, R_f and C and compute Bode Plot

$R_i = 10k$ $\quad$ $R_f = 100k$ $\quad$ $C = 0.01\ \mu F$

Adds a pole at s=0 and zero at $s = -1/R_fC$

Matlab Code

```
ri=input('Enter value of input resistance: ');
c=input('Enter value of capacitance: ');
rf=input('Enter value of feedback resistance: ');

% compute transfer function model parmeters for
% PI controller

% compute numerator parameter
tau=rf*c;
% compute parameter for denominator
tau1 = ri*c;

% build transfer function
% denominator form a1*s^2+a2s+a3
Av=tf([tau 1],[tau1 0])

%plot graph
bode(Av);
```



Integral action below 1000 rad/sec
1/R_fC sets break point

Example: Design a PI OP AMP controller with K_p = 100 and an integral break frequency of 100 rad/sec. R_{in} = 10kΩ

$\frac{R_f}{R_{in}} = K_p$ so $K_p := 100$ $R_{in} := 10000$

$R_f := R_{in} \cdot K_p$ $R_f = 1 \cdot 10^6$ Ohms

$\omega := 1000$ rad/sec

$\omega = \frac{1}{R_f \cdot C}$ so $C := \frac{1}{R_f \cdot \omega}$ $C = 1 \cdot 10^{-9}$ F

Derivative Control Mode

This mode is never used alone. Only produces an output when error is changing

Use with proportional or proportional and integral

Gives and output that is proportional to the rate of change in the error signal

Anticipates the error by observing rate of change

Ideal Derivative Mode Equations

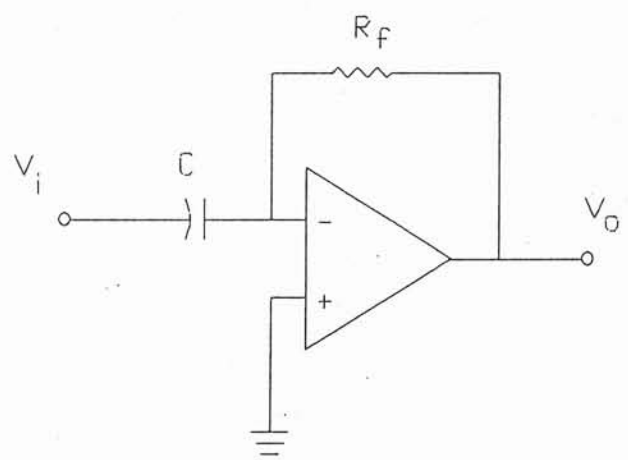
Time Function $$v(t) = K_D \cdot \frac{d}{dt} e(t)$$

Laplace $$V(s) = K_D \cdot s \cdot E(s)$$

Transfer Function $$\frac{V(s)}{E(s)} = K_D \cdot s$$

OP AMP Realizations of Differentiators

Ideal OP AMP differentiator



Transfer Function

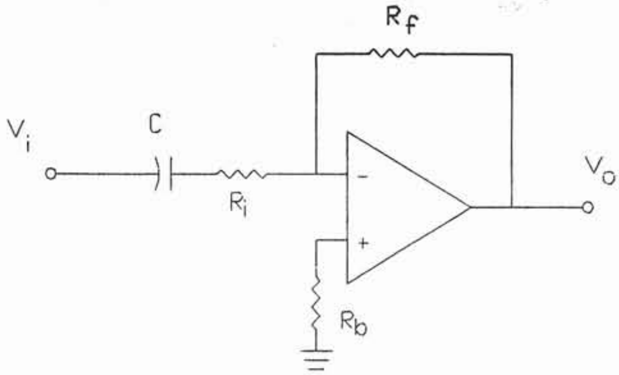
$$A_v(s) = -R_f \cdot C \cdot s$$

Introduces 1 zero at $s = 0$

Differentiators add a zero to the controller that uses them.

Differentiators are highpass filters to sine signals. Increase sensitivity to rapid changes.

Practical Differentiator Circuit



Transfer Function

$$A_v(s) = \frac{-R_f \cdot C \cdot s}{1 + R_i \cdot C \cdot s}$$

Introduces

Pole $\quad s = -\frac{1}{R_i \cdot C}$

Zero $\quad s = 0$

Pole restricts the high frequency response of the differentiator circuit

Gain and Phase Plot of Practical OP AMP Differentiator

$$A_v(\omega) = \frac{-R_f \cdot C \cdot j \cdot \omega}{\left(1 + R_i \cdot C \cdot j \cdot \omega\right)}$$

$$\left|A_v(\omega)\right| = \frac{R_f \cdot C \cdot \omega}{\sqrt{1 + R_i^2 \cdot C^2 \cdot \omega^2}}$$

$$\phi(\omega) = 90 - \operatorname{atan}\left(R_i \cdot C \cdot \omega\right) \cdot \frac{180}{\pi}$$

Proportional-Derivative Control Mode

Combines proportional and Derivative actions
Used in processes that have sudden load changes that proportional only can not handle.

Equations

Time Function

$$v(t) = K_p \cdot e(t) \ldots$$
$$+ K_p \cdot K_D \cdot \frac{d}{dt} e(t) - \alpha \cdot K_D \cdot \frac{d}{dt} v(t) + v_o$$

Note:

$\alpha \cdot K_D \cdot \frac{d}{dt} v(t)$ is a rate limit to high frequency changes

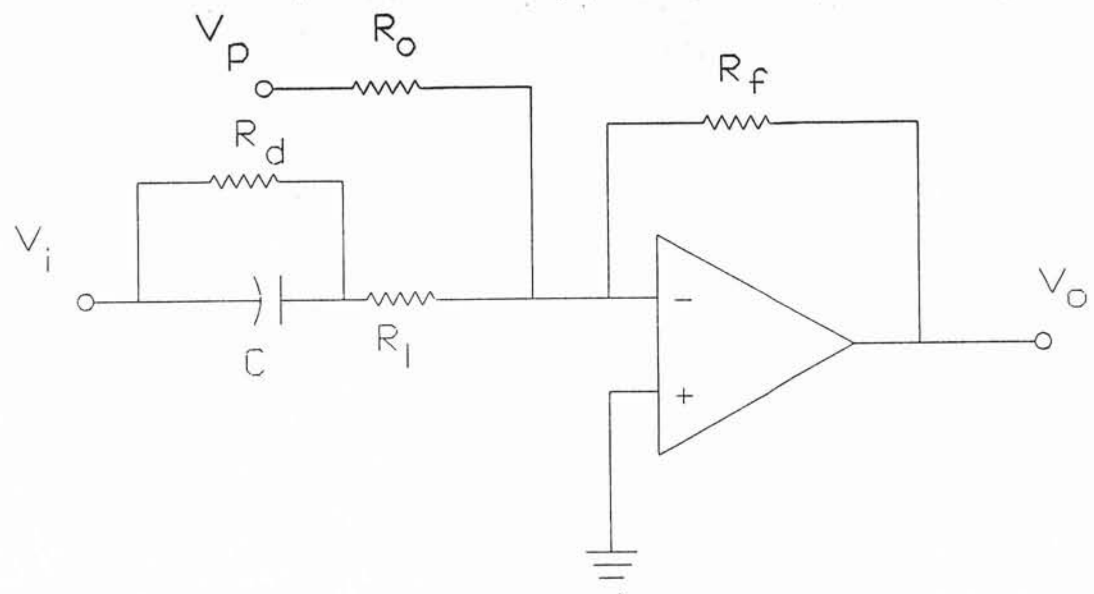
Laplace

$$V(s) = K_p \cdot E(s) + K_p \cdot K_D \cdot s \cdot E(s) - \alpha \cdot K_D \cdot s \cdot V(s)$$

Transfer Function

$$\frac{V(s)}{E(s)} = K_p \cdot \frac{1 + K_D \cdot s}{1 + \alpha \cdot K_D \cdot s}$$

OP AMP Realization of PD Controller



$$\frac{V(s)}{E(s)} = -K_p \cdot \frac{1 + K_D \cdot s}{1 + \alpha \cdot K_D \cdot s}$$

$$K_D = R_d \cdot C \qquad \alpha = \frac{R_I}{R_I + R_d} \qquad K_P = \frac{R_f}{R_I + R_d}$$

$$R_o = R_f$$

Note: $1/K_d$ = the derivative action break point f

$1/\alpha K_d$ = limiter action break point f

$0<\alpha<1$

Example: Design a PD control that has a proportional gain of 10 a derivative action breakpoint of 100 rad/sec and a limiter frequency breakpoint of 1000 rad/sec. C = 0.1 μF.

$K_p = 10 \quad \frac{1}{K_D} = 100 \text{ rad/sec} \quad \frac{1}{\alpha K_D} = 1000 \text{ rad/sec}$

$R_d C = K_D$ so $\frac{1}{R_d C} = \frac{1}{K_D} \quad C = 0.1\times 10^{-6} \quad \frac{1}{R_d(0.1\times 10^{-6})} = \frac{1}{100}$

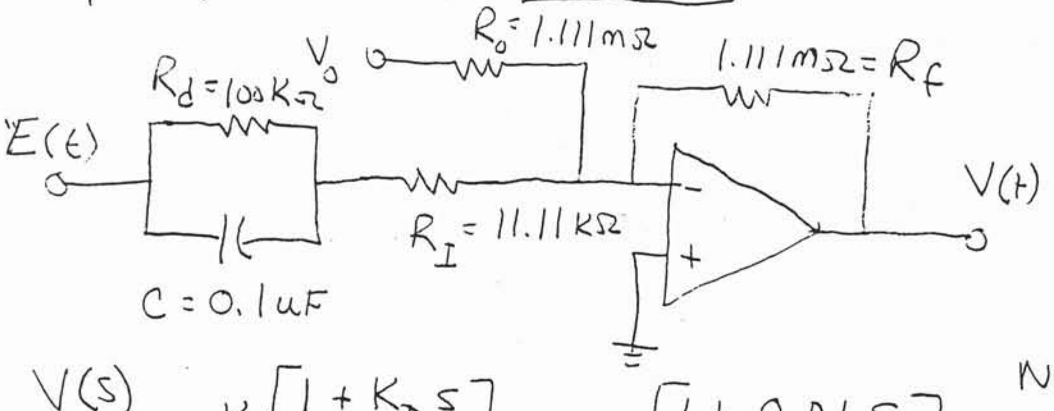
$R_d = \frac{1}{(0.1\times 10^{-6})(100)} = \boxed{100\,K\Omega} \quad \frac{1}{\alpha K_D} = 1000 \text{ rad/sec} \quad K_D = 0.01$

$\alpha = \frac{1}{0.01(1000)} = \boxed{0.1} \quad \alpha = \frac{R_I}{R_I + R_d} \Rightarrow 0.1 = \frac{R_I}{R_I + 100\,K\Omega}$.

$0.1 R_I + 10000 = R_I \Rightarrow 10{,}000 = 0.9 R_I \rightarrow \boxed{11{,}111\,\Omega = R_I}$

$\frac{R_f}{R_I + R_d} = K_p$ Substitute Values $\frac{R_f}{11{,}111 + 100\times 10^3} = 10$

$R_f = 10(11{,}111 + 100\times 10^3) = \boxed{1.111\,M\Omega} \quad R_f = R_o = 1.111\,M\Omega$



$\frac{V(s)}{E(s)} = -K_p\left[\frac{1 + K_D s}{1 + \alpha K_D s}\right] = -10\left[\frac{1 + 0.01 s}{1 + 0.001 s}\right]$ Negative sign causes 180 phase shift

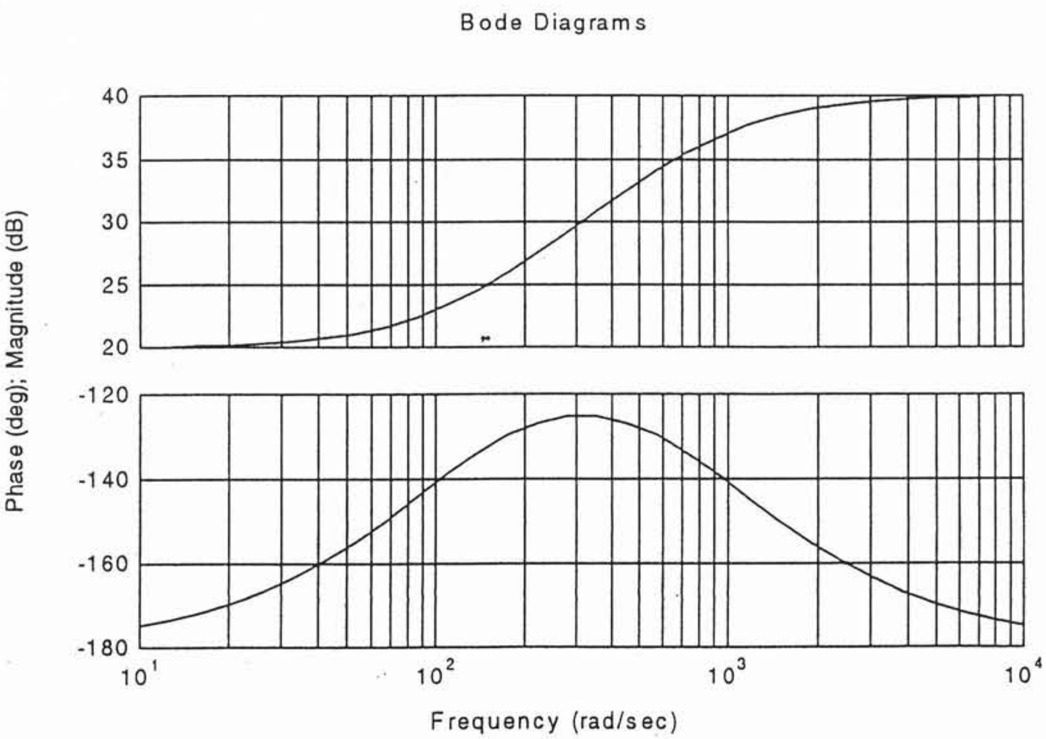
Use Matlab to produce bode plots

```
ri=11.11e3;
c=0.1e-6;
rf=1.111e6;
rd=1e5;
% compute transfer function model parmeters for
% PD controller

% compute parameters
kd=rd*c;
alpha=ri./(ri+rd);
kp=-rf./(ri+rd);
% build transfer function
% denominator form a1*s^2+a2s+a3
% numerator for b1*s^2+b2s+b3
Av1=kp*tf([kd 1],[alpha*kd 1])
%plot graph
bode(Av1);
```

Transfer function:

-0.09999 s - 9.999

0.0009999 s + 1



Proportional + Integral + Derivative Controllers (PID controller)

Proportional mode - fast response

Integral mode - removes steady-state (offset) error

Derivative mode - improves stability and improves response.

Time function

$$v = K_p \cdot e(t) + K_p \cdot K_I \cdot \int_0^t e(t)\,dt \ldots$$

$$+ K_P \cdot K_D \cdot \frac{d}{dt} e(t) - \alpha \cdot K_D \cdot \frac{d}{dt} v(t) + v_o$$

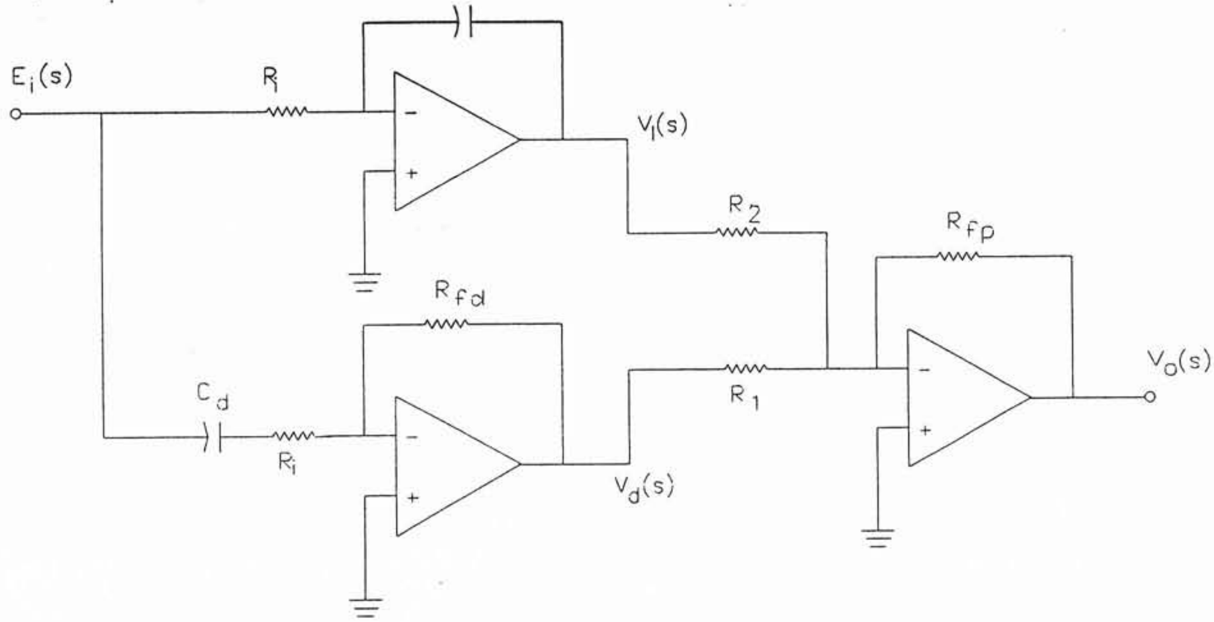
Laplace Function

$$V(s) = K_P \cdot \left[\begin{array}{l} E(s) + \dfrac{K_I}{s} \cdot E(s) + \left(K_D \cdot s \cdot E(s)\right) \ldots \\ + -\alpha \cdot K_D \cdot s \cdot V(s) \end{array} \right]$$

Transfer Function

$$\frac{V(s)}{E(s)} = K_P \cdot \left(\frac{K_I + s + K_D \cdot s^2}{s + \alpha \cdot K_D \cdot s^2} \right)$$

OP AMP Realization of the PID Controller Requires Multiple OP AMPS



$$\frac{V_I(s)}{E(s)} = -\frac{1}{C_i \cdot R_i \cdot s} \qquad \frac{V_d(s)}{E(s)} = -\frac{R_{fd} \cdot C_d \cdot s}{1 + C_d \cdot R_d \cdot s}$$

$$\frac{V_I(s)}{E(s)} + \frac{V_d(s)}{E(s)} = -\left(\frac{1}{C_i \cdot R_i \cdot s} + \frac{R_{fd} \cdot C_d \cdot s}{1 + C_d \cdot R_d \cdot s}\right)$$

$$\frac{V_o(s)}{E(s)} = \left(\frac{1}{C_i \cdot R_i \cdot s} + \frac{R_{fd} \cdot C_d \cdot s}{1 + C_d \cdot R_d \cdot s}\right) \cdot \frac{R_f}{R_1}$$

$$R_1 = R_2$$

Simplify and combine like terms

$$\frac{V_o(s)}{E(s)} = \left(\frac{1 + R_d \cdot C_d \cdot s + R_{fd} \cdot C_d \cdot R_i \cdot C_i \cdot s^2}{R_i \cdot C_i \cdot s + R_d \cdot C_d \cdot s^2} \right) \cdot \frac{R_f}{R_1}$$

Define

$$K_p = \frac{R_f}{R_1} \qquad \frac{1}{K_I} = R_i \cdot C_i \qquad K_D = R_{fd} \cdot C_d$$

$$\alpha \cdot K_D = R_d \cdot C_d$$

Make substitutions and simplify

$$\frac{V_o(s)}{E(s)} = \left(\frac{K_I + \alpha \cdot K_D \cdot K_I + K_D \cdot s^2}{s + \alpha \cdot K_D \cdot K_I \cdot s^2} \right) \cdot K_p$$

This controller add two zeros and two poles to the transfer function of the system. The location of these points depends on the gain values of controller

Can change response of system by adjusting values of K_p K_D, K_I and α

Design a PID controller using the circuit above. Proportional gain is 5. The derivative time constant is 0.5 seconds. The integral gain is 0.143 and α=0.1. The capacitor for the integrator C_i=10 μF and the differentiator, C_d= 1 μF. R_{fp} = 1 MΩ. Find values for all other components.

$K_I = \frac{1}{R_i C_i}$ $C_i = 10\,\mu F$

$R_i = \frac{1}{K_I C_i} = \frac{1}{(0.143)(10\mu F)} = 700,000\,\Omega$

$R_i = 700,000\,\Omega$

$K_p = 5$ $R_{fp} = 1\,M\Omega$

$\frac{R_{fp}}{R_1} = K_p \Rightarrow R_1 = \frac{R_{fp}}{K_p}$

$R_1 = \frac{1\times 10^6}{5} = 200,000\,\Omega$

$R_1 = R_2 = 200,000\,\Omega$

$K_D = 0.5$ $K_d = R_{fd} C_d$

$C_d = 1\,\mu F$

$\frac{K_d}{C_d} = R_{fd}$ $\frac{0.5}{1\times 10^{-6}} = 500,000\,\Omega = R_{fd}$ $\alpha = 0.1$

$\alpha K_D K_I = R_d C_d$ $C_d = 1\,\mu F$

$\frac{\alpha K_D K_I}{C_d} = R_d$ substitute values

$\frac{0.1(0.5)(0.143)}{1\times 10^{-6}} = R_d$

$7150\,\Omega = R_d$