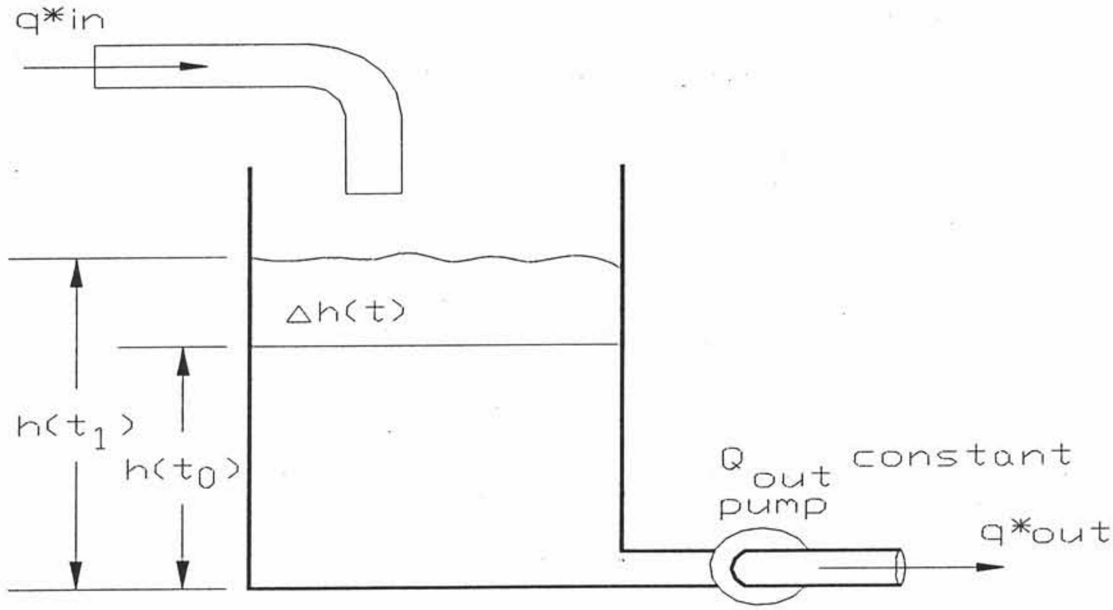
Process Characteristics

Integral Process - single energy storage element (capacitance)

Example -Tank with constant out flow



Time domain equation

$$h(t_1) - h(t_0) = \frac{1}{T_i} \cdot \int_{t_0}^{t_1} (q_{in} - q_{out})\, dt$$

Transfer function

$$\frac{H(s)}{Q_{in}(s)} = \frac{1}{T_i \cdot s}$$

where $h(t_0)$ = normalized output at t_0 percentage of FS_{out}

$h(t_1)$ = normalized output at t_1 percentage of FSout

FS_{in} = full-scale range of the input

FS_{out} = full-scale range of output

q_{in} = normalized input flow rate as percentage of FS_{in}

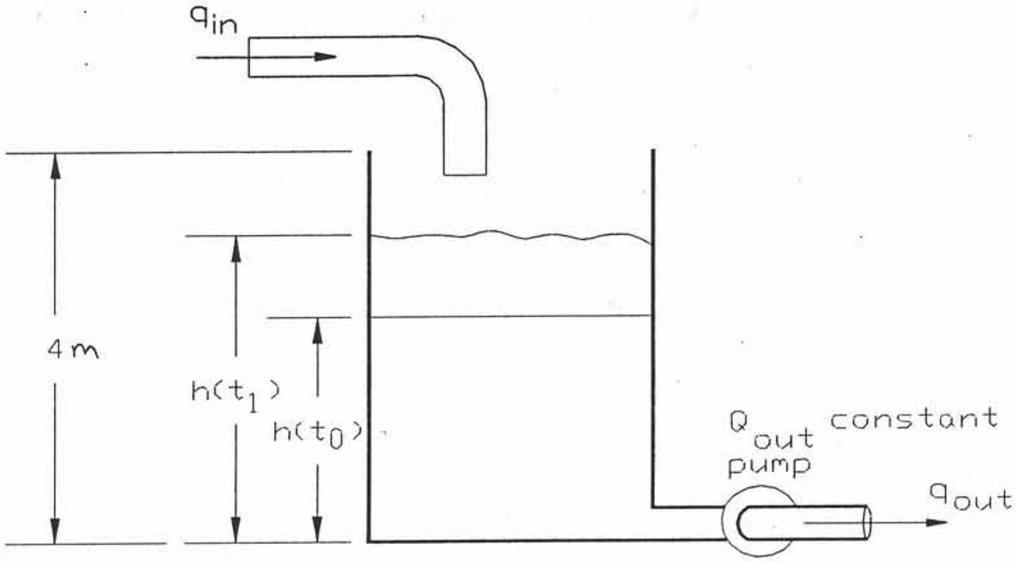
q_{out} = normalized output flow rate as percentage of FS_{out}

T_i = integral action time constant

integral constant $$T_i = A \cdot \frac{FS_{out}}{FS_{in}}$$

A = cross-sectional area of tank at liquid surface m^2.

Example:



For tank system shown, find the transfer function when tank diameter = 1.5 m and h = 4 m. Full scale q_{in} = 0.01 m^3/s

$$A = \frac{\pi d^2}{4} = \frac{\pi (1.5\,m)^2}{4} = 1.77\,m^2 \qquad FS_{in} = q_{in} = 0.01\,m^3/s$$

$$FS_{out} = 4\,m$$

$$h(t_0) = 22.5\%\,FS \qquad 4\,m\left(\frac{22.5\%}{100\%}\right) = 0.9\,m$$

$$q_{out} = 60\%\,FS \qquad (0.01\,m^3/s)\left(\frac{60\%}{100\%}\right) = 0.006\,m^3/s$$

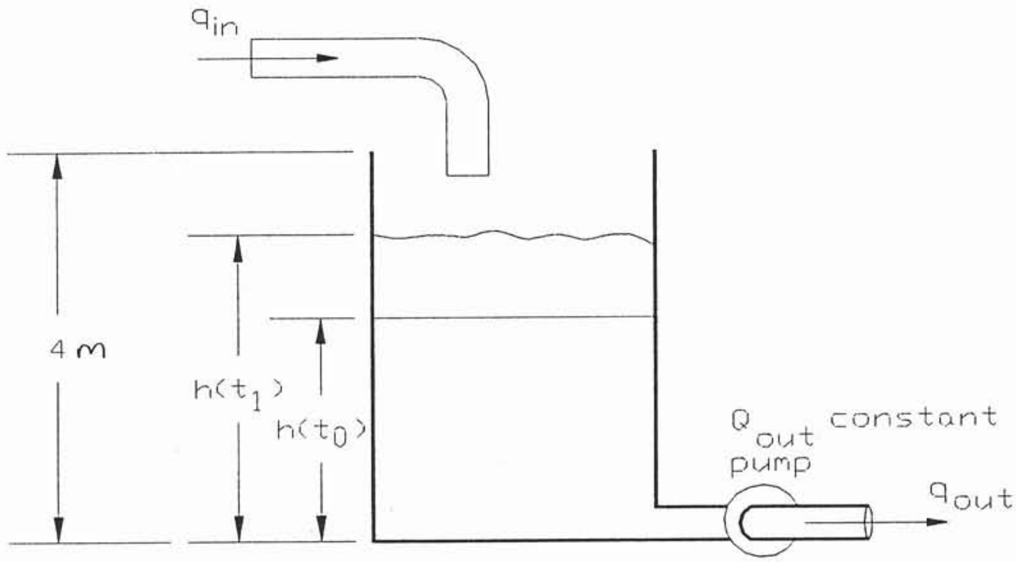
$$q_{in} = 80\%\,FS \qquad (0.01\,m^3/s)(80\%/100\%) = 0.008\,m^3/s$$

$$T_{in} = A\frac{FS_{out}}{FS_{in}} = \frac{1.77\,m^2(4\,m)}{0.01\,m^3/s} = 703\,s$$

$$\boxed{\frac{H(s)}{Q_{in}(s)} = \frac{1}{T_i s} = \frac{1}{703\,s}}$$

Example: Continued

Determine the time domain equation, and the tank level when after 100 seconds has elapsed from t_0.



$$h(t_1) - h(t_0) = \frac{1}{T_i}\int_{t_0}^{t_1}(q_{in} - q_{out})\,dt \qquad t_1 = 100\text{ s} \quad h(t_0) = 22.5\%\ (0.9\text{ m})$$

$$h(t_1) = \frac{1}{T_i}\int_{t_0}^{t_1}(q_{in} - q_{out})\,dt + h(t_0) \qquad T_{in} = 703\text{ s}$$

$q_{in} = 80\%\ (0.008\ m^3/s) \quad q_{out} = 60\%\ (0.006\ m^3/s)$

$$h(100) = \frac{1}{703}\int_0^{100}(80-60)\,dt + 22.5\% = \frac{1}{703}\left[20t\Big|_0^{100}\right] + 22.5\%$$

$$h(100) = \frac{1}{703}\left[20(100)\right] + 22.5\% = 2.85\% + 22.5\% = \boxed{25.34\%}$$

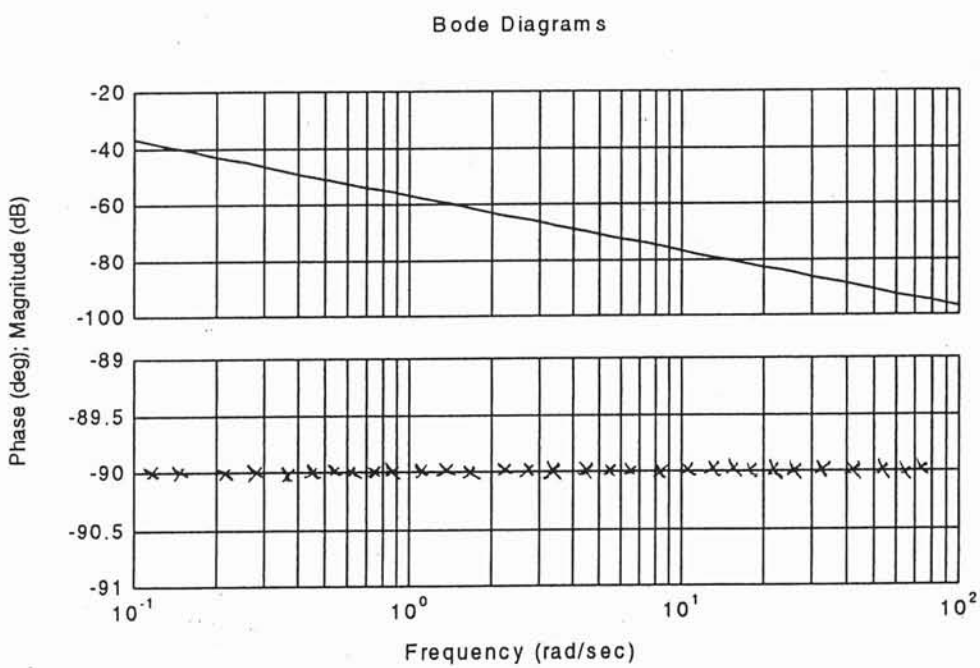
convert to actual height

$$\frac{25.34\%}{100\%}FS_{out} = h(t) = \frac{25.34\%}{100\%}(4\text{ m}) = \boxed{1.014\text{ m}}$$

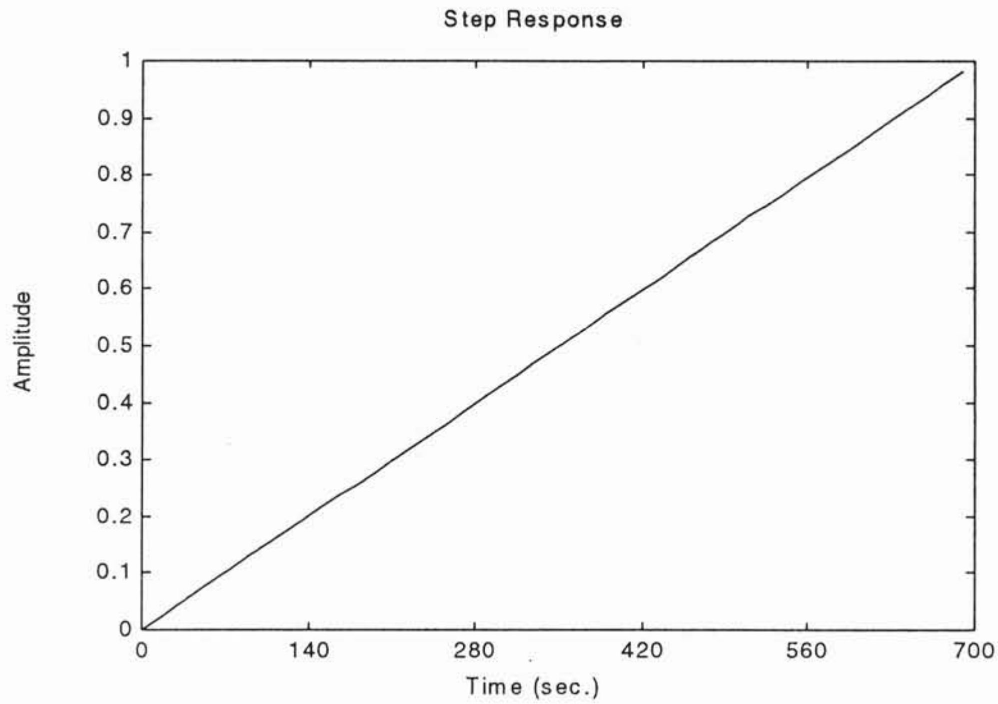
Step Response and Bode Plots
Integral Process

Matlab code

```
% close all previous figures and clear all variables
close all;
clear all;
% input the integral time constant
Ti=input('enter the integral time constant: ');
% construct and display the system
sys=tf(1,[Ti 0]);
sys
% plot the frequency response
bode(sys);
% construct a new figure and plot the time response
figure;
% define a range of time
t=(0:15:Ti);
% use it to generate a step response plot
step(sys,t);
```



Plot the time response - note that output reaches 100% of full scale when T_i =703 sec elapses.



First-Order Lag Process

Characteristics: Single storage element
Input produces an output related to amount of storage

Self-regulating process

Examples: Series R-C circuit, Series R-L circuit
Self-regulating tank (valve on output)
Tank heating

First-Order Lag Process General Model

Time-domain equation

$$\tau \cdot \frac{d}{dt} y + y = G \cdot x$$

Transfer Function

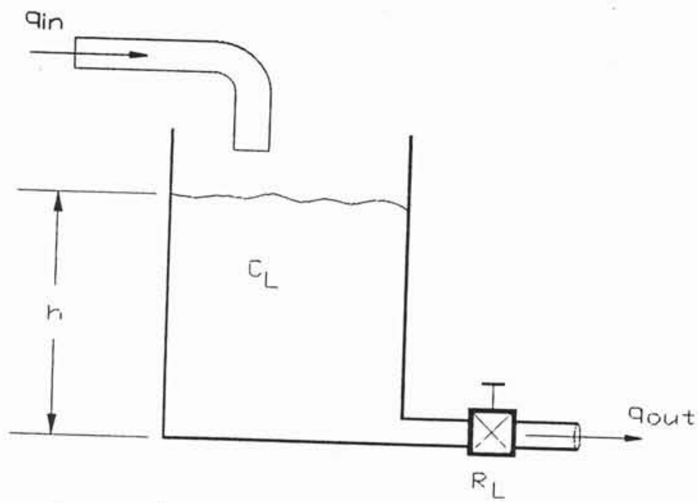
$$\frac{Y(s)}{X(s)} = \frac{G}{1 + \tau \cdot s}$$

Where: G = steady-state gain of the system
t = time, sec
y = output of process (units or %FS)
x = input of process (units or %FS)
τ = time constant of the system, sec

The value of τ determines the time a system requires to reach final value after step input change

After 1τ output 63.2% of final value
After 5τ output 99.3% of final value

Self-Regulating Tank



Process input: q_{in} = input flow rate (% FS_{in})

FSin = input range (m^3/s)

Process ouput: h = tank level, (% FS_{out})

FS_{out} = output range (m)

Time constant: $\tau = R_L C_L$ where R_L is the flow resistance and C_L is the tank capacitance

Steady-state gain

$$G = \frac{R_L}{\rho \cdot g} \cdot \frac{FS_{in}}{FS_{out}}$$

Where r = liquid density kg/m^3

g = acceleration due to gravity (m/s^2)

Example: Oil tank similar to previous figure has a diameter of 1.25 m and a height of 2.8 m. The outlet pipe is a smooth tube with a length of 5 m and diameter of 2.85 cm. Oil temperature 15 degrees C. The full scale flow rate is 24 L/min and full scale height is 2.8 m. Determine: a.) tank capacitance b.) pipe resistance, c.) process time constant d.) process gain, e.) time-domain equation, f.) transfer function.

a.) $C_L = \frac{A}{\rho g}$ $d = 1.25\,m$ $\rho = 880\,Kg/m^3$ from appendix

$$A = \frac{\pi d^2}{4} = \frac{\pi (1.25\,m)^2}{4} = 1.23\,m^2$$

$$C_L = \frac{1.23\,m^2}{880\,Kg/m^3 (9.81\,m/s^2)} = \boxed{1.425 \times 10^{-4}\ m^3/Pa}$$

b.) Find liquid resistance. Compute Reynold's number

$\mathcal{R} = \frac{\rho v d}{\mu}$ v = average velocity

μ = absolute viscosity use max Q to find $\mathcal{R}$

$Q = VA$

$$24\ L/min \left(1.6667 \times 10^{-5} \frac{m^3/s}{L/min}\right) = 4 \times 10^{-4}\ m^3/s \qquad V = \frac{Q}{A} \qquad A = \frac{\pi (2.85 \times 10^{-2} m)^2}{4}$$

$\rho = 880\ kg/m^3$ $d = 2.85$ cm pipe dia.

$\mu = 0.160$ Pa-s $A = 6.3794 \times 10^{-4}\ m^2$

$$\mathcal{R} = \frac{(880\ Kg/m^3)(0.627\ m/s)(2.85 \times 10^{-2} m)}{0.160\ Pa\text{-}S} \qquad V = \frac{4 \times 10^{-4}\ m^3/sec}{6.3794 \times 10^{-4}\ m^2}$$

$V = 0.627$ m/sec

$$\mathcal{R} = \frac{15.725}{0.160} = 98.28$$

Example: (cont.) $R < 2000$ Laminar flow

for Laminar Flow $R_L = \frac{128 \mu l}{\pi d^4}$ $Pa \cdot s/m^3$ $R_L = \frac{128(0.160)(5)}{\pi (2.85\times10^{-2} m)^4}$

$l = 5 m$ $\mu = 0.160$ Pa-s

$R_L = \frac{102.4}{\pi\, 6.5975\times10^{-7}} = \boxed{4.94\times10^{7}\ Pa\text{-}s/m^3}$

c.) Process Time Constant $\tau = R_L C_L$

$\tau = (4.94\times10^{7}\ Pa\text{-}s/m^3)(1.425\times10^{-4}\ m^3/Pa) = \boxed{7040\ s}$

$\tau = 7040\ s\left(\frac{1\ min}{60\ s}\right) = \boxed{117.3\ min.}$

Tank level will go down by 63.2% after 117.3 min.
if $q_{in} = 0$ It will be 99.2% empty after 5τ

d.) Process Gain $G = \frac{R_L}{\rho g}\left[\frac{FS_{in}}{FS_{out}}\right]$ $FS_{in} = 4\times10^{-4}\ m^3/s$
$FS_{out} = 2.8$ m
$\rho = 880\ Kg/m^3$
$R_L = 4.94\times10^{7}\ Pa\text{-}s/m^3$

$G = \frac{4.94\times10^{7}\ Pa\text{-}s/m^3}{880\ Kg/m^3 (9.81\ m/s^2)}\left[\frac{4\times10^{-4}\ m^3/s}{2.8 m}\right]$

$G = 5722.36(1.4286\times10^{-4}) = 0.818$

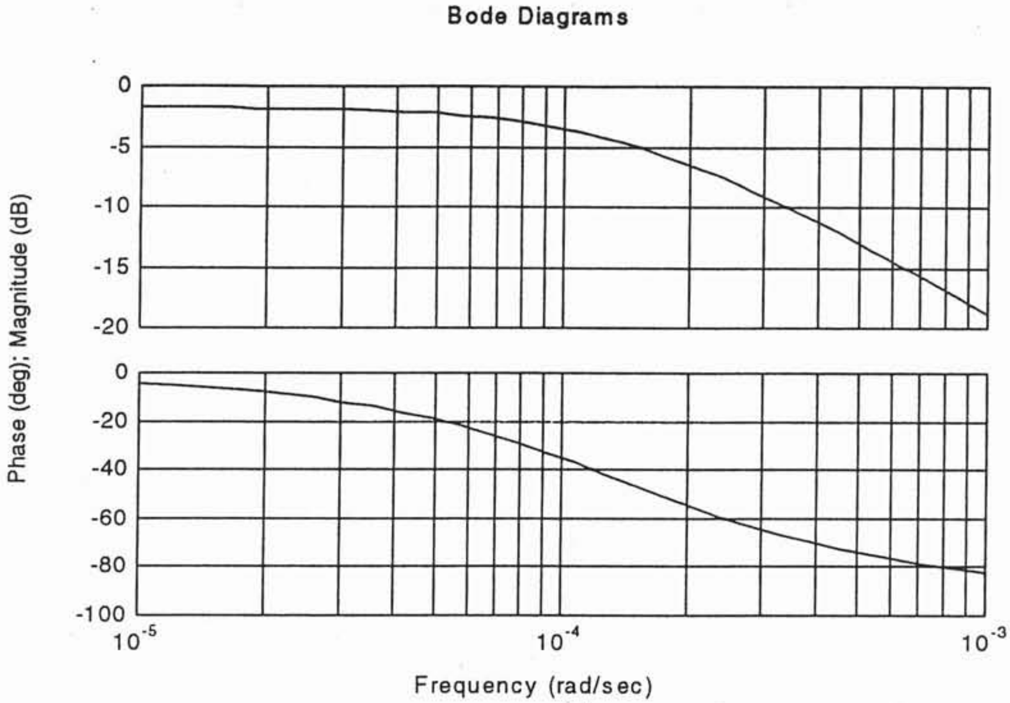
e.) Basic form $\tau \frac{dy}{dt} + y = Gx$ $x = q_{in}$ $y = h$
q_{in} and h are % FS

$\tau = 7040 s$
$7040\frac{dh}{dt} + h = 0.818\, q_{in}$

f.) Basic form of Transfer function $\frac{H(s)}{Q_{in}(s)} = \frac{G}{1+\tau s}$

$$\frac{H(s)}{Q_{in}(s)} = \frac{0.818}{1+7040 s}$$

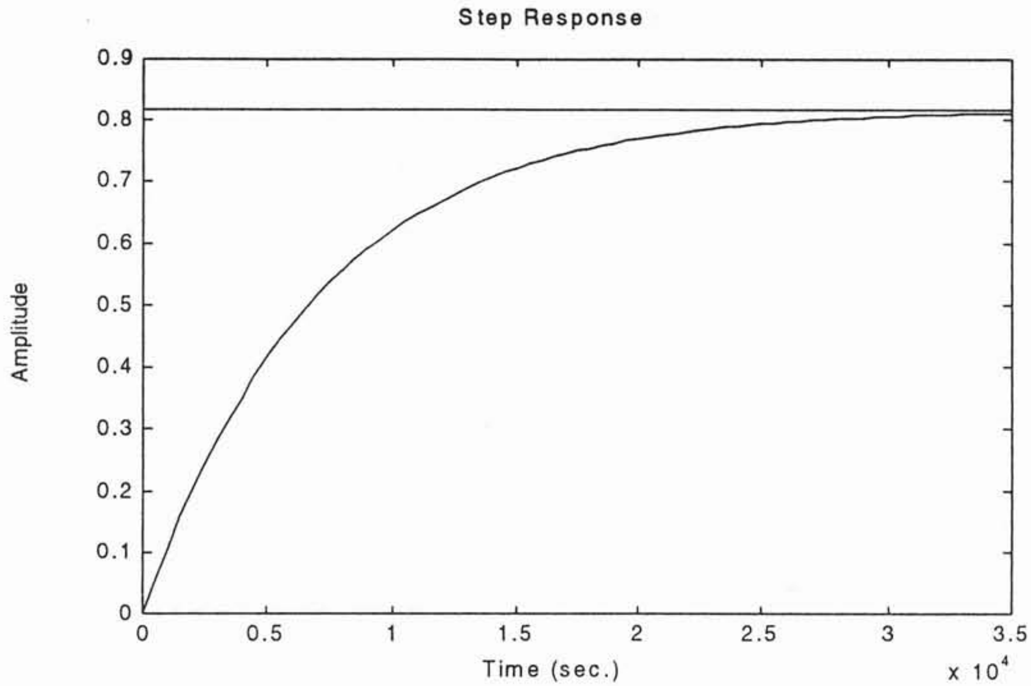
Bode Plots and Step Responses of Tank System



Matlab code

```
% close all previous figures and clear all variables
close all;
clear all;
% input the integral time constant
Ti=input('enter the integral time constant: ');
G=input('enter the gain of the process: ');
% construct and display the system
sys=tf(G,[Ti 1]);
sys
% plot the frequency response
bode(sys);
% construct a new figure and plot the time response
figure;
% define a range of time
t=(0:500:5*Ti);
% use it to generate a step response plot
step(sys,t);
```

Step response



5τ gives the final output value $\tau := 7040 \cdot sec$

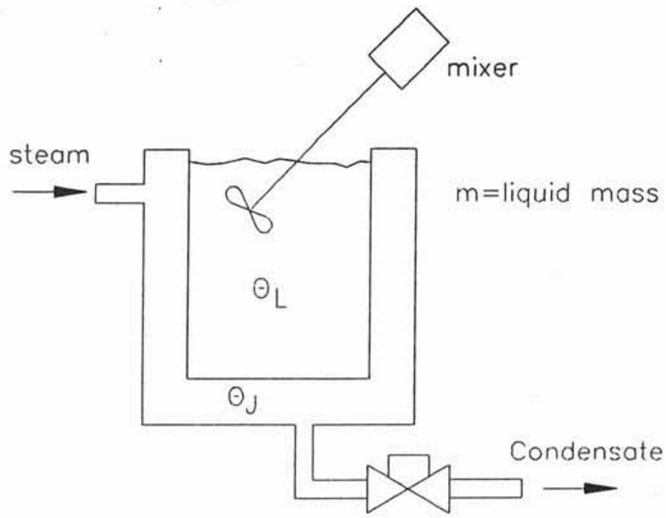
$$5 \cdot \tau = 3.52 \cdot 10^{4} \cdot sec$$

From bode plot note the cutoff frequency, f_c

When gain is down 3 db $f=f_c$ These are very low f's

$$f_c := \frac{1}{\tau} \qquad f_c = 1.42 \cdot 10^{-4} \cdot \frac{rad}{sec}$$

Thermal lag-process example



Temperature of oil bath θ_L depends on the steam temperature θ_J and the thermal resistance and capacitance of the system

Basic model $$R_T \cdot C_T \cdot \frac{d}{dt}\theta_L + \theta_L = \theta_J$$

Assumes that steam flow and temperature is constant The above tank is filled with oil. It is 1.2 m tall and has a diameter of 1 m. The inside film coefficient is 62 W/m^2-K and the outside film coefficient is 310 W/m^2-K. The tank is made of steek with a wall thickness of 1.2 cm. Find a.) thermal resistance b.) thermal capacitance, c.) thermal time constant d.) differential equation model, e.) transfer function model.

Thermal lag system solution

$R_{u_A} = \frac{1}{h_i} + \frac{X}{K_s} + \frac{1}{h_o}$ unit thermal R

a.) K_s = thermal coefficient of steel

$K_s = 45$ W/m-K° X = wall thickness = 1.2 cm = <u>0.012 m</u>

h_i 62 W/m²-K° h_o = 310 W/m²-K°

$$R_u = \frac{1}{62\ W/m^2\text{-}K^\circ} + \frac{0.012\ m}{45\ W/m\text{-}K} + \frac{1}{310\ W/m^2\text{-}K^\circ}$$

$R_u = (0.01613 + 2.667\times10^{-4} + 0.00323)\ m^2K/W$

$R_u = 0.01962\ m^2\text{-}K/W$ $R_T = \frac{R_u}{A}$ Area is total area of tank. area of bottom $A_1 = \frac{\pi D^2}{4} = \frac{\pi(1\ m)^2}{4}$ area of wall area of cylinder

$A_T = A_1 + A_2$ $A_1 = 0.7854\ m^2$ $A_2 = \pi D h = \pi(1)(1.2m)$

$A_T = 0.7854 + 3.7699\ m^2$ $A_2 = 3.7699\ m^2$ $h = 1.2\ m$

$A_T = 4.5553\ m^2$

$$R_T = \frac{R_u}{A} = \frac{0.01962\ m^2\text{-}K/W}{4.5553\ m^2} = \boxed{0.004307\ K/W}$$

b.) $C_T = m S_h$ S_h = heat capacity of oil S_h = 2180 J/Kg-K Appendix A
m = mass of oil find from density and volume of tank.

$\rho = 880\ Kg/m^3$ Appendix A

$$Vol = \frac{\pi d^2}{4} h = \frac{\pi(1m)^2}{4}(1.2m) = 0.9425\ m^3$$ $m = \rho(Vol)$

$m = 880\ Kg/m^3(0.9425\ m^3)$

$m = 829.38\ Kg$

$C_T = (829.38\ Kg)(2180\ J/Kg\text{-}K)$

$\boxed{C_T = 1.808\times10^6\ J/K}$

Thermal lag system solution (cont.)

$\tau = R_T C_T = (0.004307\ K/W)(1.808 \times 10^6\ J/K)$

$\tau = 7787\ s$ (129.8 min) Unit step change in input takes 5τ to settle out

$5\tau = 649$ mins 10.82 Hrs

d.) $\tau \frac{dy}{dt} + y = x$ $x = \theta_J$ $y = \theta_L$ $\tau = 7787 s$

$$7787 \frac{d\theta_L}{dt} + \theta_L = \theta_J$$

e.) $\frac{\theta_L(s)}{\theta_J(s)} = \frac{1}{1 + \tau s}$ $G = 1$

$$\frac{\theta_L(s)}{\theta_J(s)} = \frac{1}{1 + 7787 s}$$

Second-Order Lag Processes

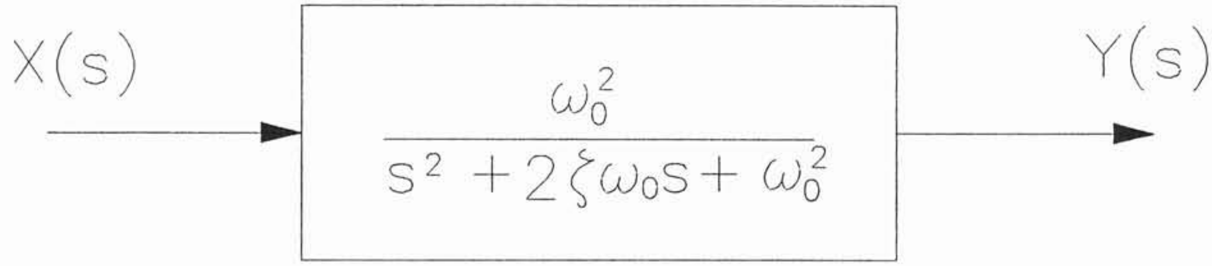
Characteristics - contain two energy storage elements (examples: 2 capacitances, 1 mass and 1 spring, 1 capacitance and 1 inductance etc.)

System response determined by three parameters

steady-state gain, G damping ratio ζ and resonant frequency, ω_0.

Second Order Systems

Systems that have only two dominate poles (two closest to imaginary axis) can be approximated by second order system.



Transfer Function using standard notation for a second order system

ω_0 = the natural frequency of the system (resonance)

ζ = the damping coefficient of the system

These values are determined by the system parameters

Characteristic equation in terms of the parameters ω_0 and ζ

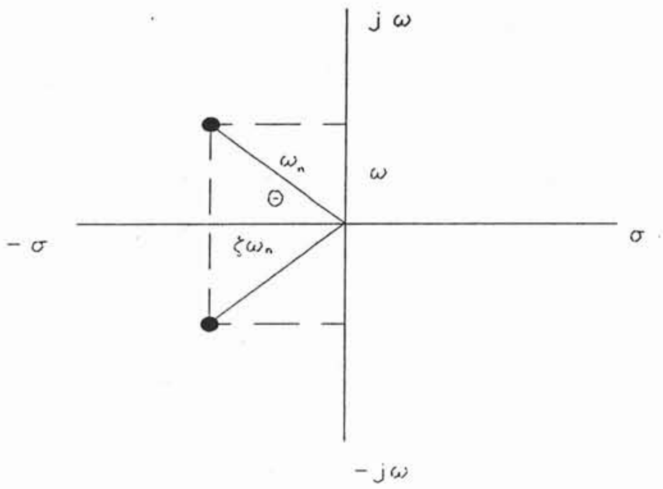
$$s^2 + 2\cdot\zeta\cdot\omega_0\cdot s + \omega_0^2 = 0$$

Roots

$$s_1 = -\zeta\cdot\omega_0 + j\cdot\omega_0\cdot\sqrt{\zeta^2 - 1} \qquad s = \sigma + j\cdot\omega$$

$$s_2 = -\zeta\cdot\omega_0 - j\cdot\omega_0\cdot\sqrt{\zeta^2 - 1} \qquad s = \sigma - j\cdot\omega$$

Plot of roots on complex plane. As poles approach $j\omega$ axis the system becomes more oscillatory.



$$\zeta \cdot \omega_0 = \sigma$$

This controls the exponential rise and decay

If $0 < \zeta < 1$ then

$$\omega = \omega_0 \cdot \sqrt{\zeta^2 - 1}$$

Called <u>conditional frequency</u>, not sinusoidal but damped sinusoidal

If $\zeta = 0$, damping is zero system will oscillate at $\omega = \omega_0$

Roots of quadratic formula can have three possible forms

1.) real - distinct
2.) real - identical
3.) imaginary - conjugate pairs

Location of roots is controlled by the values of ζ and ω_0.
If natural frequency is constant them damping controls system response

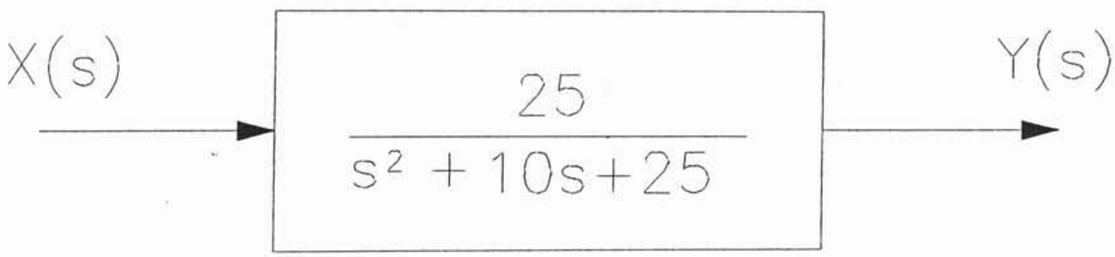
Effects of changing the damping coefficient on the response of a second order system.

$\zeta = 1$ - system is called critically damped. Reaches the final value the fastest without having any overshoot. Roots are equal and real

$\zeta < 1$ - system is called underdamped. Reaches the final value the fast but with overshoot. Less damping more overshoot. Roots are conjugate pair.

$\zeta > 1$ - system is called overdamped. Reaches the final value the slowly but with no overshoot. More damping slower response to final value. Roots are real but not equal.

Example 1: Transfer Function



Find ω_0 and ζ for this system and determine if the system is critically, under or over damped

Equate coefficients

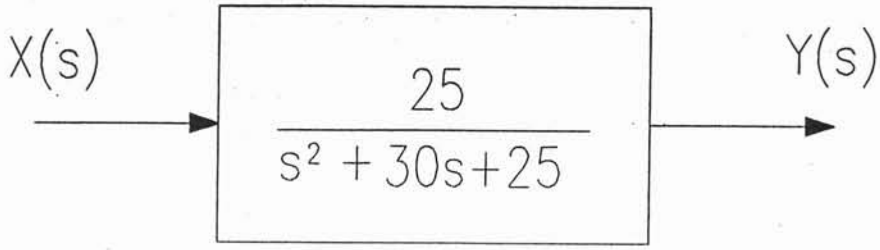
$$s^2 + 10 \cdot s + 25 = s^2 + 2 \cdot \zeta \cdot \omega_0 \cdot s + \omega_0^2$$

$$\omega_0^2 = 25 \qquad \text{so} \qquad \omega_0 = 5$$

$$2 \cdot \zeta \cdot \omega_0 = 10 \qquad 2 \cdot \zeta \cdot 5 = 10 \qquad \zeta = 1$$

system is critically damped

Example 2: transfer function



Equate coefficients $s^2 + 30 \cdot s + 25 = s^2 + 2 \cdot \zeta \cdot \omega_0 \cdot s + \omega_0^{\,2}$

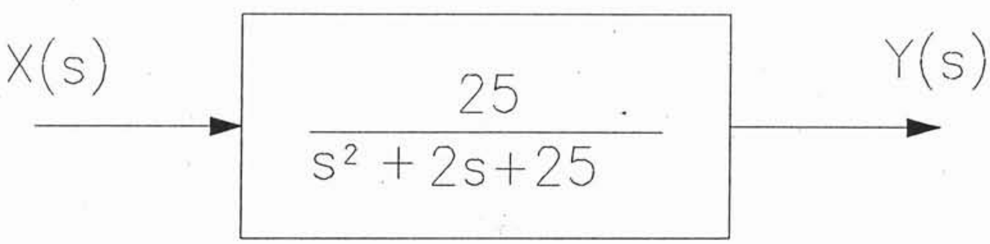
$\omega_0^{\,2} = 25$ so $\omega_0 = 5$

$2 \cdot \zeta \cdot \omega_0 = 30$ $\quad 2 \cdot \zeta \cdot 5 = 30$ $\quad \zeta = 3$

system is overdamped damped

As damping increases one of the poles becomes more dominant than the other. The dominant pole controls the overall response.

Example 3: transfer function

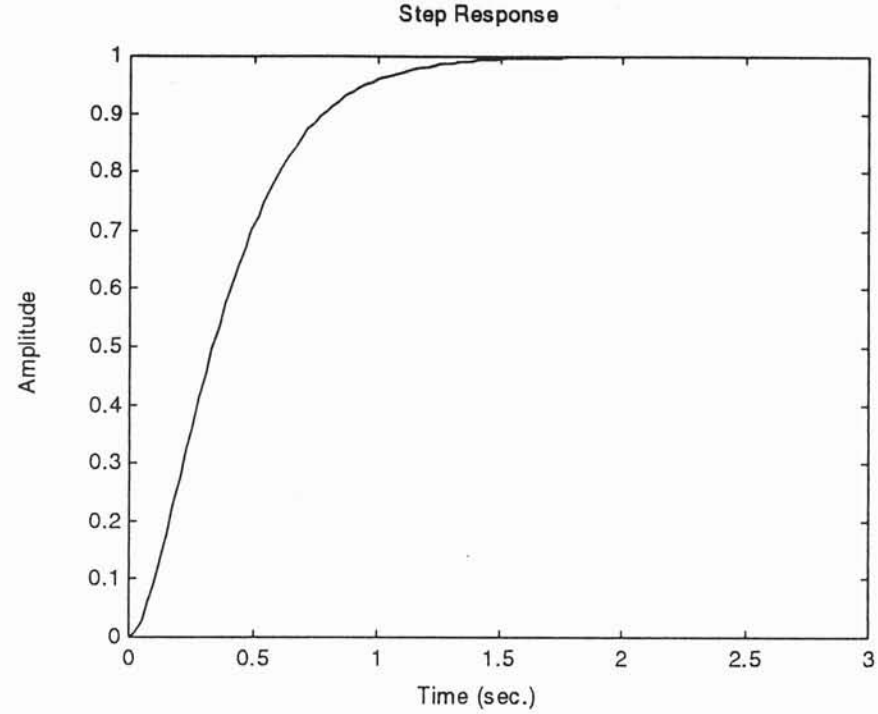


Equate coefficients $s^2 + 2 \cdot s + 25 = s^2 + 2 \cdot \zeta \cdot \omega_0 \cdot s + \omega_0^{\,2}$

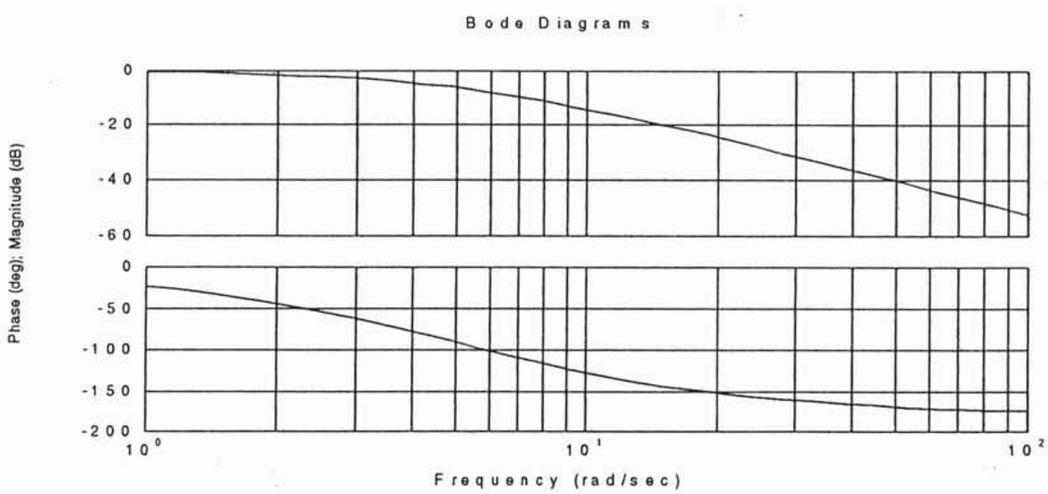
$\omega_0^{\,2} = 25$ so $\omega_0 = 5$

$2 \cdot \zeta \cdot \omega_0 = 2$ $\quad 2 \cdot \zeta \cdot 5 = 2$ $\quad \zeta = 0.2$

Critically damped System

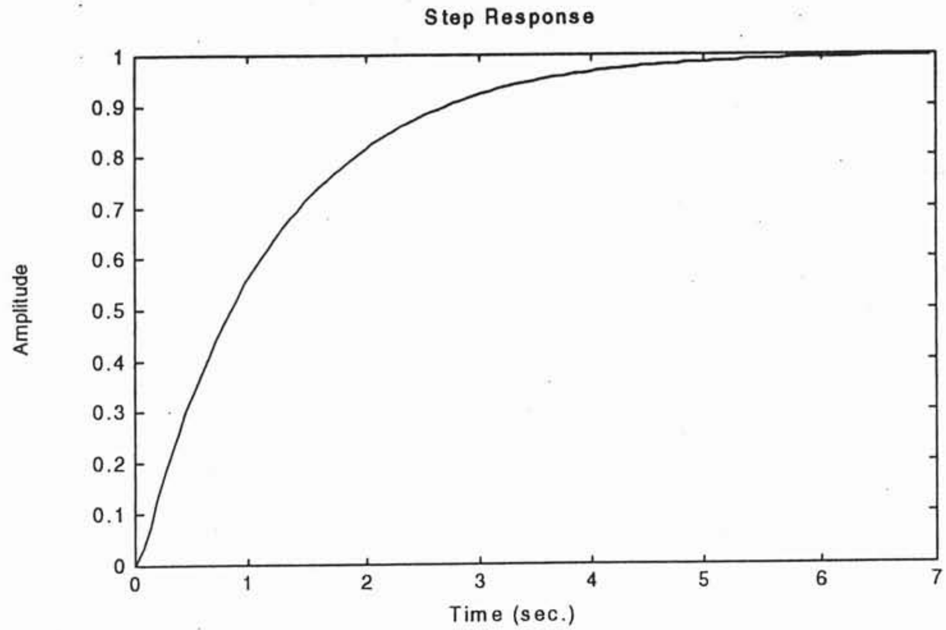


Output reaches the final value in approximately 1.4 s. No overshoot

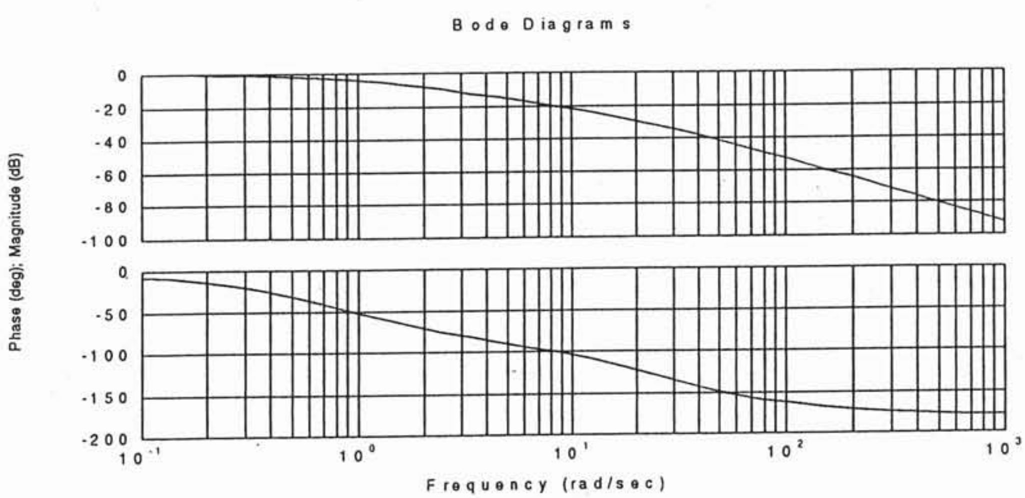


Acts like a lag process

Over Damped System

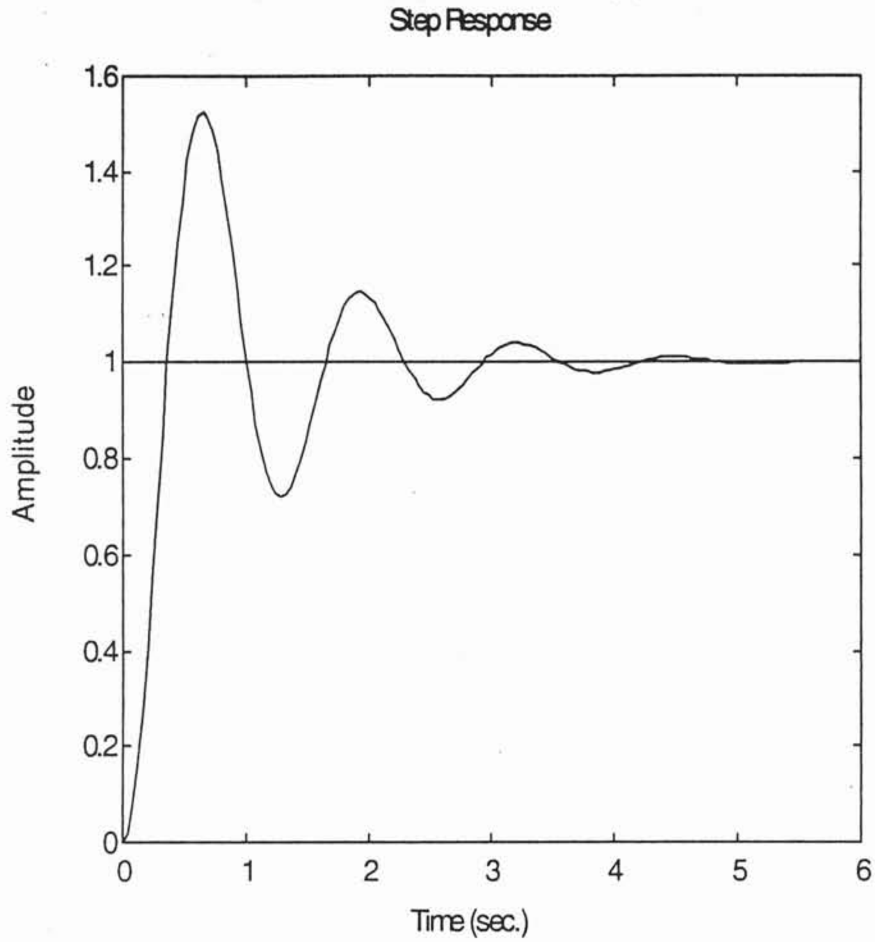


Compare to the critically damped case, the response time is slower. Approx. 6.5s to get to final value.



Slower response than critically damped system

Underdamped System

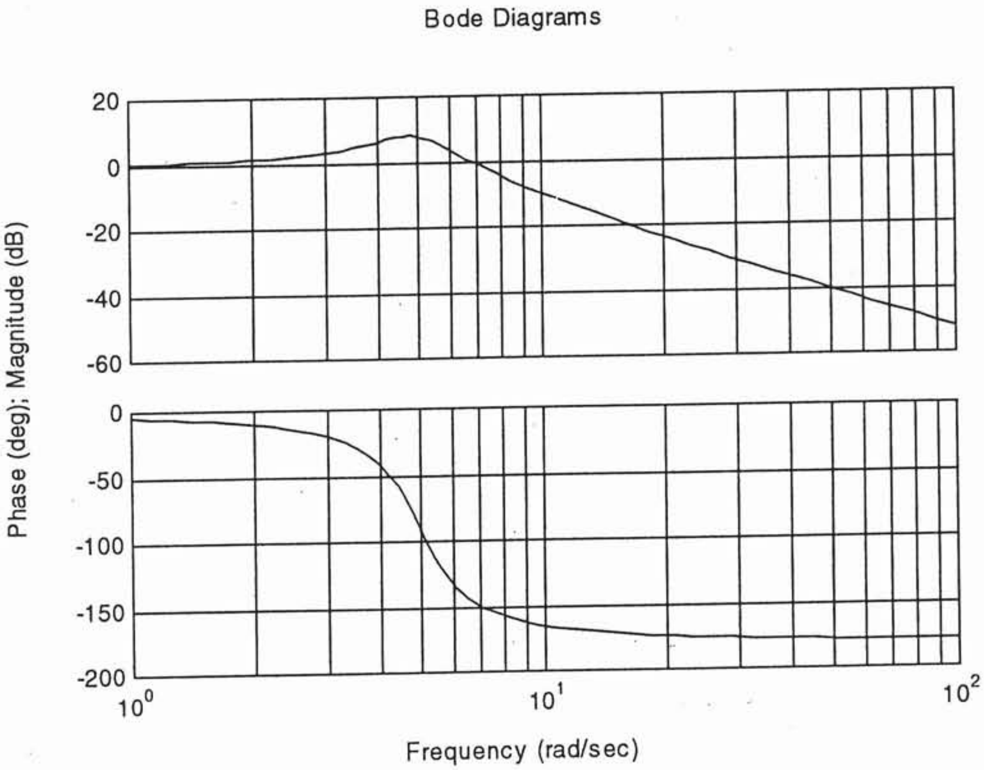


Notice: Overshoot in the system. Max value approx. 1.5
Settling time can be estimated in this plot. Approx. 4.5s.

Generate a Bode plot for comparison to the others

Bode plot of underdamped system.

Resonant peak starting to form at the natural frequency of the system.. $\omega_0 = 5$ rad/s



This shows a resonant peak at the resonant frequency

Note phase plot is different in each case also

General 2nd Order Lag Process Equations

Time domain

$$A_2 \cdot \frac{d^2}{dt^2} y + A_1 \cdot \frac{d}{dt} y + y = G \cdot x$$

Transfer Function

$$\frac{Y(s)}{X(s)} = \frac{G}{1 + A_1 \cdot s + A_2 \cdot s^2}$$

Parameters

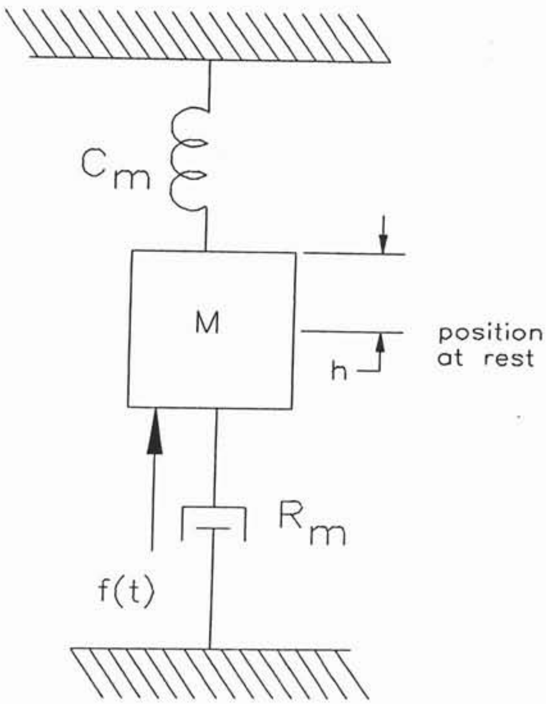
$$\omega_0 = \sqrt{\frac{1}{A_2}} \qquad \alpha = \frac{A_1}{2 \cdot A_2}$$

α =Damping coefficient

$$\zeta = \frac{\alpha}{\omega_0} = \frac{A_1}{2 \cdot \sqrt{A_2}} = \frac{A_1 \cdot \omega_0}{2}$$

$$A_2 = \frac{1}{\omega_0^{\,2}} \qquad A_1 = \frac{2 \cdot \zeta}{\omega_0} = \frac{2 \cdot \alpha}{\omega_0^{\,2}}$$

Example: Second order Mechanical system



Input signal: input force f(t)
Output signal: mass centerline position, h.

Newton's Law

F = ma

Sum of external forces equals the mass time the acceleration.

$$f(t) + f_s + f_d = M \cdot \frac{d^2}{dt^2} h$$

C_m = capacitance of the spring (inverse of spring constant)
R_m = resistance due to viscous friction
M = mass

$$f_s = \frac{-h}{C_m}$$

f down when position above reference
f up when position below reference

$$f_d = -R_m \cdot \frac{d}{dt} h$$

f down when velocity moving up
f up when velocity moving down

$$f(t) - \frac{h}{C_m} - R_m \cdot \frac{d}{dt}h = M \cdot \frac{d^2}{dt^2} h$$

Arrange above into a 2nd order differential equation

$$f(t) = M \cdot \frac{d^2}{dt^2} h + R_m \cdot \frac{d}{dt}h + \frac{h}{C_m}$$

$$C_m \cdot M \cdot \frac{d^2}{dt^2} h + C_m \cdot R_m \cdot \frac{d}{dt}h + h = f(t) \cdot C_m$$

Referring the above to the general form for the 2nd order system

$A_2 = C_m \cdot M \qquad A_1 = C_m \cdot R_m \qquad G = C_m$

Find the coefficients if the Mass-Spring-Damper system has the following values M = 10 kg, R_m = 20 N-s/m, and C_m=0.001 m/N.

Also find: a.) the transfer function of the system
b.) the resonant frequency
c.) the damping ratio
d.) determine if the process is overdamped, underdamped or critically damped.

$M := 10 \cdot kg \qquad R_m := 20 \cdot \frac{N \cdot sec}{m} \qquad C_m := 0.001 \cdot \frac{m}{N}$

$A_2 := M \cdot C_m \qquad A_1 := R_m \cdot C_m \qquad G := C_m$

$A_2 = 0.01 \cdot sec^2 \qquad A_1 = 0.02 \cdot sec \qquad G = 1 \cdot 10^{-3} \cdot \frac{m}{N}$

Substitute into differential equation

$$0.01 \cdot \frac{d^2}{dt^2} h + 0.02 \cdot \frac{d}{dt} h + h = f(t) \cdot 0.001 \qquad \underline{\text{ANS}}$$

a.) Find transfer function

Take Laplace transform of both sides of equation

$$0.01 \cdot s^2 \cdot H(s) + 0.02 \cdot s \cdot H(s) + H(s) = F(s) \cdot 0.001$$

$$\left(0.01 \cdot s^2 + 0.02 \cdot s + 1\right) \cdot H(s) = 0.001 \cdot F(s)$$

Output H(s) input F(s)

$$\frac{H(s)}{F(s)} = \frac{0.001}{\left(0.01 \cdot s^2 + 0.02 \cdot s + 1\right)} \qquad \textbf{\underline{ANS}}$$

b.) Find the resonant f

$$\omega_0 := \sqrt{\frac{1}{A_2}} \qquad A_2 = 0.01 \cdot sec^2$$

$$\omega_0 = 10 \cdot sec^{-1} \cdot rad \qquad (rad/sec)$$

c.) Find the damping ratio

$$\zeta := \frac{A_1}{2 \cdot \sqrt{A_2}} \qquad \zeta = 0.1$$

d.) The damping ratio $\zeta<1$ so the system is underdamped. When a force is applied the mass center will oscillate before coming to rest at a new position.

Change the value of R_m to 100 N-sec/m and recompute damping ratio

$$R_m := 100 \cdot \frac{N \cdot sec}{m} \qquad A_1 := C_m \cdot R_m \qquad A_1 = 0.1 \cdot sec$$

$$\zeta := \frac{A_1}{2 \cdot \sqrt{A_2}} \qquad \zeta = 0.5$$

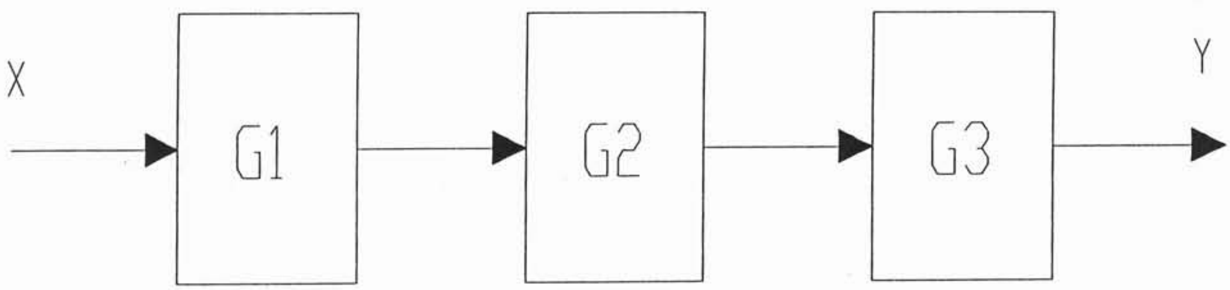
R_m effects the damping.
More losses in system
less chance of oscillations

Methods Of System Analysis

Use Bode plots of systems to determine frequency response of controller and process combined.

Replace s in transfer functions with jω and plot gain and phase shift

Bode plots of cascaded stages: multiply gains, add phase angles



$\frac{Y}{X} = G1 \cdot G2 \cdot G3$ magnitudes of each stage at a given frequency ω.

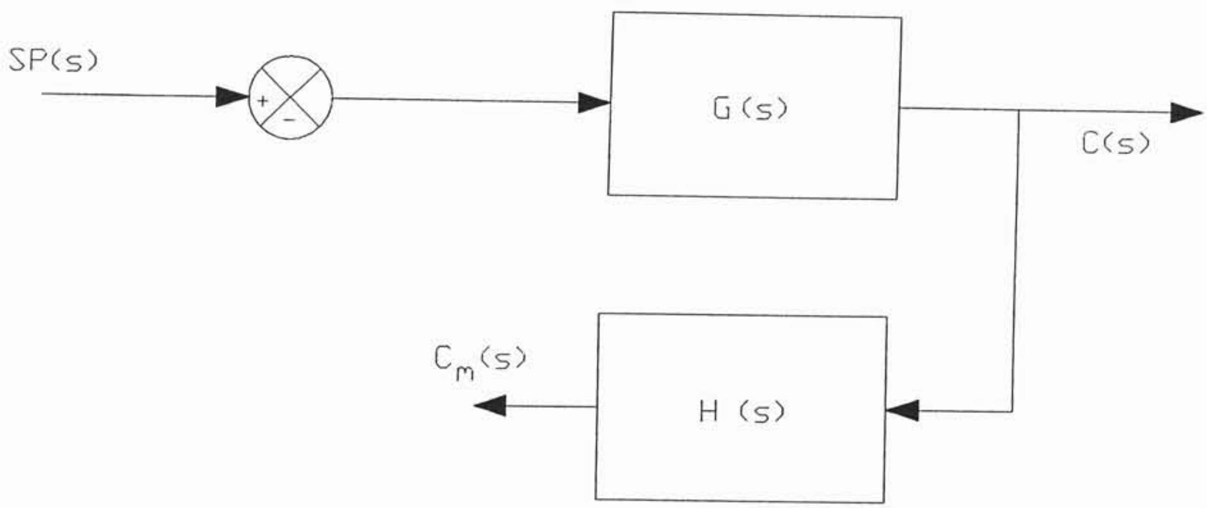
$\beta_T = \beta_1 + \beta_2 + \beta_3$ Phase shifts of each stage at a given frequency ω.

Limits of control determined by comparing the open loop response of system to closed loop response.

Open loop response of control.

$$\frac{C_m}{SP} = G(s) \cdot H(s)$$

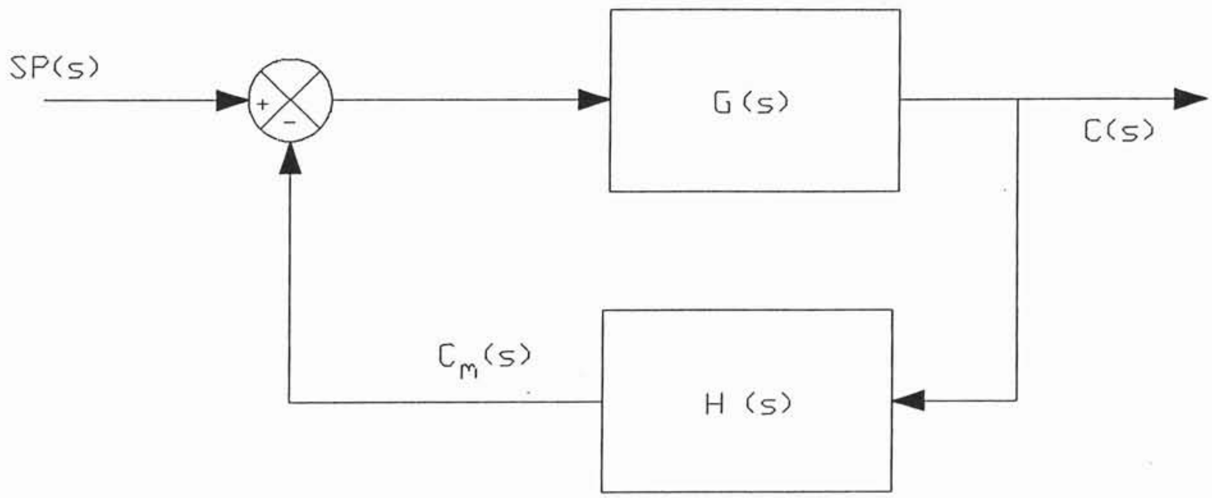
Where C_m = the measurement feedback
SP = setpoint signal
G(s) = forward path gain
H(s) = feedback path gain



In the open loop mode the error signal is equal to the setpoint value.

Open loop consists of the controller, the manipulating element, the process, and the measurement system transfer functions.

Closed loop response



Closed loop error response

$$\frac{C_m}{SP} = \frac{G(s)\cdot H(s)}{1 + G(s)\cdot H(s)}$$

Note: this is not the I/O relationship that was used earlier. C(s)/SP(s)

Frequency response of system divided into three ranges

Zone 1: controller decreases error
Zone 2: controller increases error
Zone 3: controller has no effect on error

The frequency of the change in the setpoint determines which zone will be activated. Systems overall frequency response determines the f's where these zones occur.