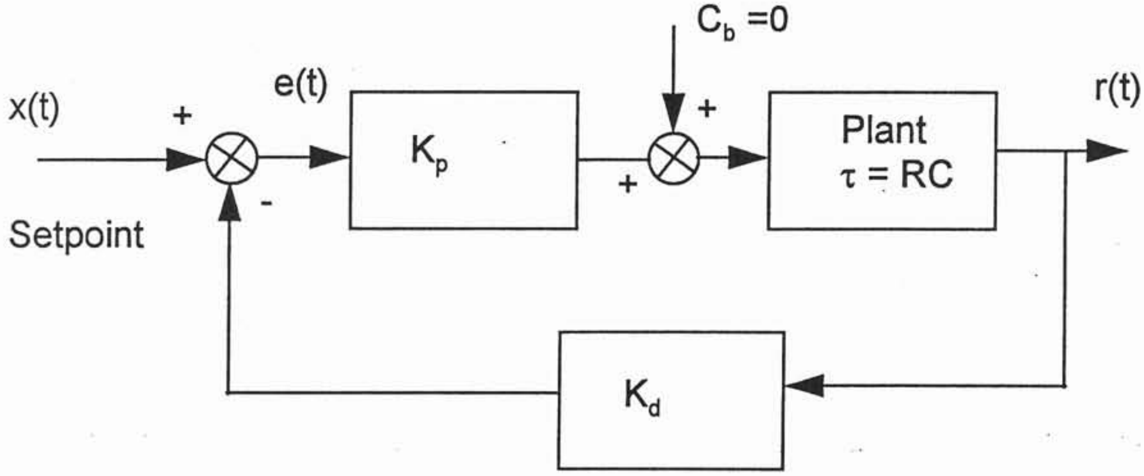
Model of Proportinal System
Bias = 0



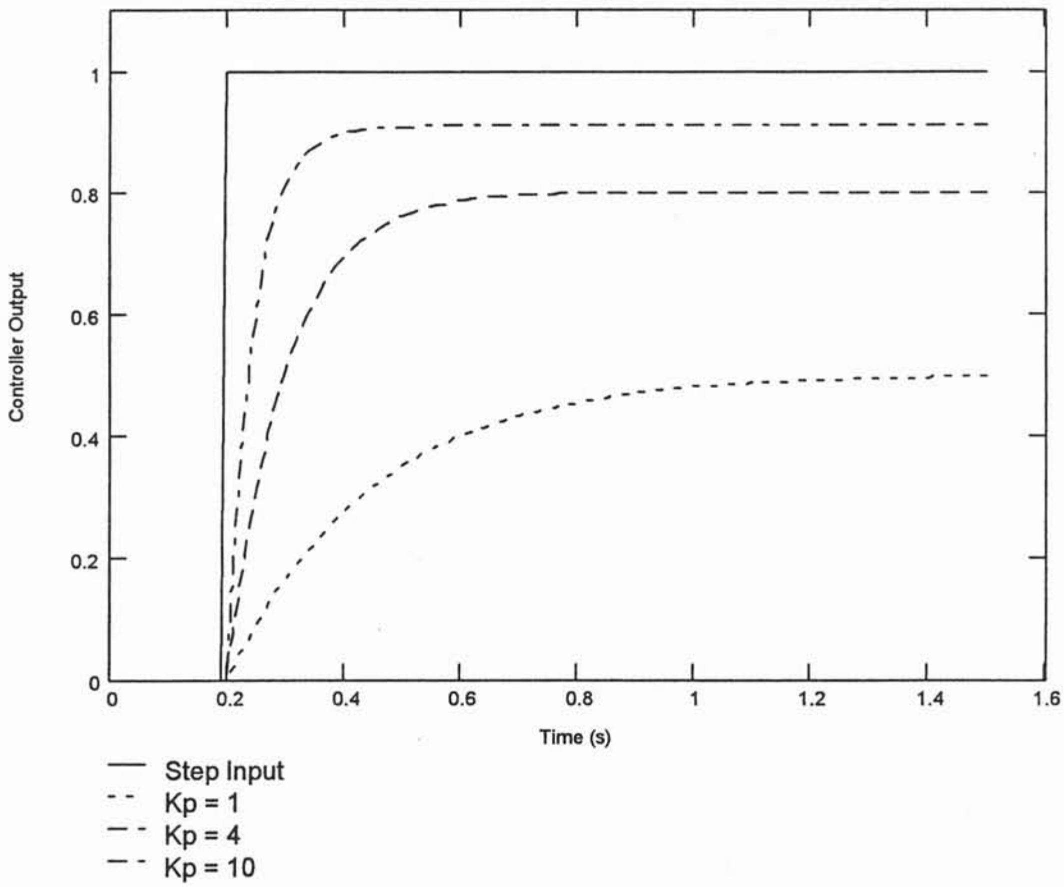
K_d = feedback gain (voltage divider in lab)
K_p = Proportional controller gain
x(t) = input function
r(t) = output response of the control system
Plant is modelled using RC circuit

Method of solution

Signal flow algebra and Laplace transforms
Final solution to step change input:

$$r(t, K_p) := \frac{K_p}{1 + K_p \cdot K_d} \cdot \left[1 - e^{\frac{-t \cdot (1 + K_p \cdot K_d)}{R \cdot C}} \right]$$

Response to Step Input



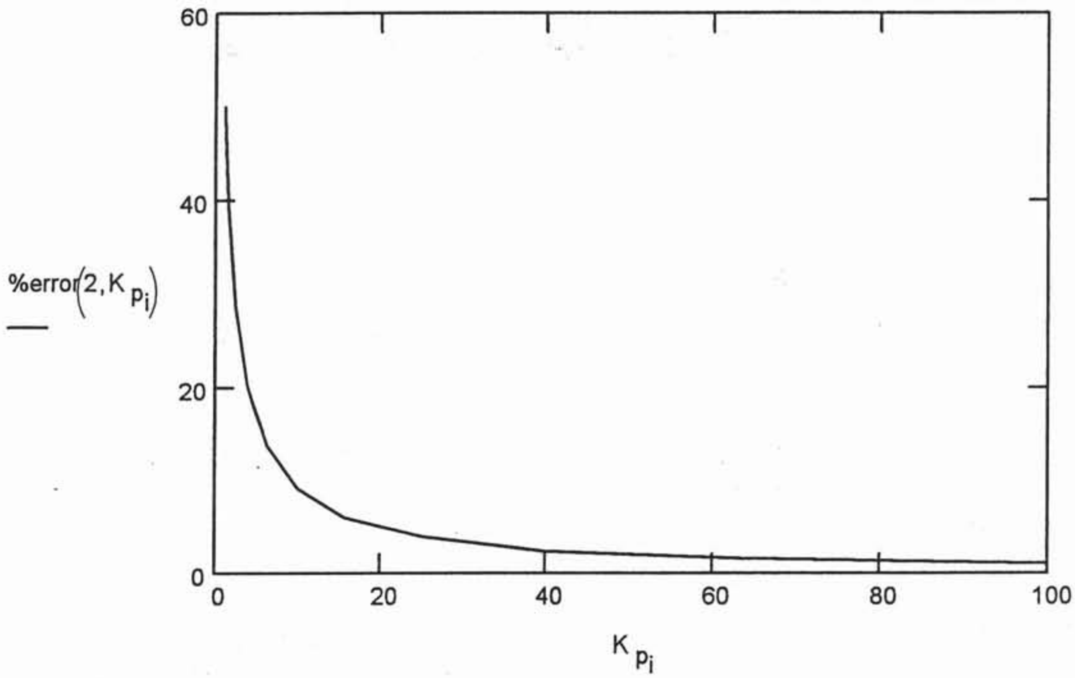
As K_p increases, steady-state error decreases
System responds faster than natural time constant

MathCAD formulas used to generate plots

$$x(t) := \Phi(t - 0.2)$$

$$r(t, K_p) := \frac{K_p}{1 + K_p \cdot K_d} \cdot \left[1 - e^{\frac{-(t-.2)\cdot\Phi(t-.2)\cdot(1 + K_p \cdot K_d)}{R \cdot C}} \right]$$

Error As a Function of Proportional Gain



Computing residual error

$$\%error = \frac{SS_d - SS_a}{SS_d} \cdot 100\%$$

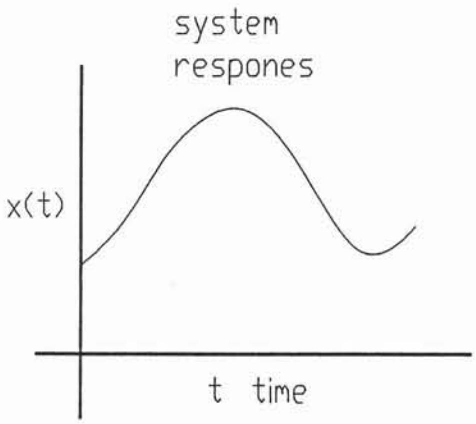
Where SS_d = desired steady-state output (1 for unit step

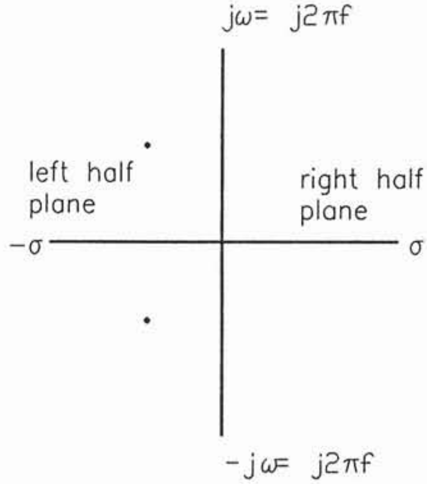
SS_a = actual output at steady-state

If K_p is plot on a log axis then this graph becomes linear

Laplace Transforms

Laplace transform converts time domain problems into functions of a complex variable, s, that is related to the frequency response of the system





Time domain Complex frequency

Complex Frequency combines transient response with sinusoidal steady-state response to get total response of system to input

$$s = \sigma + j \cdot \omega$$

σ = exponential decay/increase constant related to time constants of systems transient response RC = L/R = σ in circuit analysis

$e^{\sigma \cdot t}$ exponentially increasing function over t

$e^{-\sigma \cdot t}$ exponentially decreasing function over t

Sinusiodal response From Laplace

$j\omega = j2\pi f$ same frequency used in phasor analysis

Laplace related to sine response through Euler's Identity. Euler's relates complex exponentials to sine and cosine time functions

$$e^{j\cdot\omega} = \cos(\omega\cdot t) + j\cdot\sin(\omega\cdot t)$$

$$e^{-j\cdot\omega} = \cos(\omega\cdot t) - j\cdot\sin(\omega\cdot t)$$

Adding and subtracting the above relationships gives the exponential forms of sine and cosine

$$e^{j\cdot\omega} + e^{-j\cdot\omega} = 2\cdot\cos(\omega\cdot t)$$

$$\cos(\omega\cdot t) = \frac{e^{j\cdot\omega t} + e^{-j\cdot\omega t}}{2}$$

$$e^{j\cdot\omega t} - e^{-j\cdot\omega t} = 2\cdot j\cdot\sin(\omega\cdot t)$$

$$\sin(\omega\cdot t) = \frac{e^{j\cdot\omega t} - e^{-j\cdot\omega t}}{2\cdot j}$$

Since $e^{s\cdot t} = e^{\sigma\cdot t}\cdot e^{j\cdot\omega t}$ Laplace can give complete response

Dc, transient, steady-state sinusiodal

Basic Laplace Transform pairs

Time function	Laplace Function
$\delta(t)$ Impulse	1
$u_s(t)$ Unit step	$\frac{1}{s}$
$e^{-a \cdot t}$	$\frac{1}{s+a}$
$e^{a \cdot t}$	$\frac{1}{s-a}$
$\sin(\omega \cdot t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega \cdot t)$	$\frac{s}{s^2+\omega^2}$
t Linear ramp slope 1	$\frac{1}{s^2}$

Note: time functions multiplied by constants give Laplace function by constant

Examples: time	Laplace
$5 \cdot u_s(t)$	$\frac{5}{s}$
$3 \cdot \sin(4 \cdot t)$	$3 \cdot \frac{4}{s^2+16}$

More Laplace Examples

time	Laplace	
$e^{-2 \cdot t}$	$\frac{1}{s+2}$	
$e^{5 \cdot t}$	$\frac{1}{s-5}$	
$10 \cdot t$	$\frac{10}{s^2}$	
$t \cdot e^{-a \cdot t}$	$\frac{1}{(s+a)^2}$	Laplace Table 3.2 text
$3 \cdot \cos(t)$	$3 \cdot \frac{s}{s^2+1}$	$\omega = 1$

Theorems

Laplace of an unknown function

$$\mathcal{L}\left(f_1(t)\right) = F_1(s)$$

Examples

$$\mathcal{L}\left(i_1(t)\right) = I_1(s) \qquad \mathcal{L}\left(v_1(t)\right) = V_1(s)$$

Linearity of transform - can multiply by constant

If $\mathcal{L}\left(f_1(t)\right) = F_1(s) \qquad \mathcal{L}\left(f_2(t)\right) = F_2(s)$

Then $\mathcal{L}\left(a \cdot f_1(t) + b \cdot f_2(t)\right) = a \cdot F_1(s) + b \cdot F_2(s)$

Laplace Transforms of Calculus Operators

Laplace turns derivative into multiplication by s

If $\mathcal{L}\left(f_1(t)\right) = F_1(s)$ Then

$$\mathcal{L}\left(\frac{d}{dt} f_1(t)\right) = s \cdot F_1(s) - f_1(0)$$

Subtract any initial condition

For higher order derivatives

$$\mathcal{L}\left[\left(\frac{d^2}{dt^2} f_1(t)\right) = s \cdot \left(s \cdot F_1(s) - f_1(0)\right)\right] - \frac{d}{dt} f(0)$$

If initial conditions are all zero, formula reduces to

$$\mathcal{L}\left[\left(\frac{d^2}{dt^2} f_1(t)\right) = s \cdot \left(s \cdot F_1(s)\right)\right] = s^2 \cdot F_1(s)$$

Laplace turns integration into division by s

If $\mathcal{L}\left(f_1(t)\right) = F_1(s)$ Then

$$\mathcal{L}\left(\int f_1(t)\, dt\right) = \frac{1}{s} \cdot F_1(s)$$

Examples: Lumped circuit elements

Inductor voltage

$$v_L(t) = L\cdot\frac{d}{dt}i(t)$$

$$\mathcal{L}\, v_L(t) = \mathcal{L}\left(L\cdot\frac{d}{dt}i(t)\right) \qquad V_L(s) = L\cdot s\cdot I(s)$$

Capactior Voltage

$$v_C(t) = \frac{1}{C}\cdot\int i_c(t)\, dt$$

$$\mathcal{L}\left(v_c(t)\right) = \mathcal{L}\left(\frac{1}{C}\cdot\int i_c(t)\, dt\right) \qquad V_C(s) = \frac{1}{C\cdot s}\cdot I(s)$$

Resistor Voltage

$$v_R(t) = R\cdot i(t) \qquad \mathcal{L}\left(v_R(t)\right) = \mathcal{L}(R\cdot i(t))$$

$$V_R(s) = R\cdot I(s)$$

Similar methods can be used on lumped elements for translational and rotational mechancial systems

Laplace and Impedance

Remember phasor analysis, only valid for sinusoidal steady-state. Turns ac analysis into an analysis similar to the dc. (Ohm's law)

$$R \qquad X_C = \frac{1}{j \cdot \omega \cdot C} \qquad X_L = j \cdot \omega \cdot L \qquad \omega = 2 \cdot \pi \cdot f$$

$j = 90$ Phase shift $\qquad -j = -90$ Phase shift

Since Laplace variable represents frequency, it's possible to replace jω with s and s with jω. If s is replaced with jω, analysis becomes reverts to phasors We can find the frequency response of a dynamic system by converting differential equation into Laplace domian and replacing s with jω. Sweeping frequency produces Bode plot of system

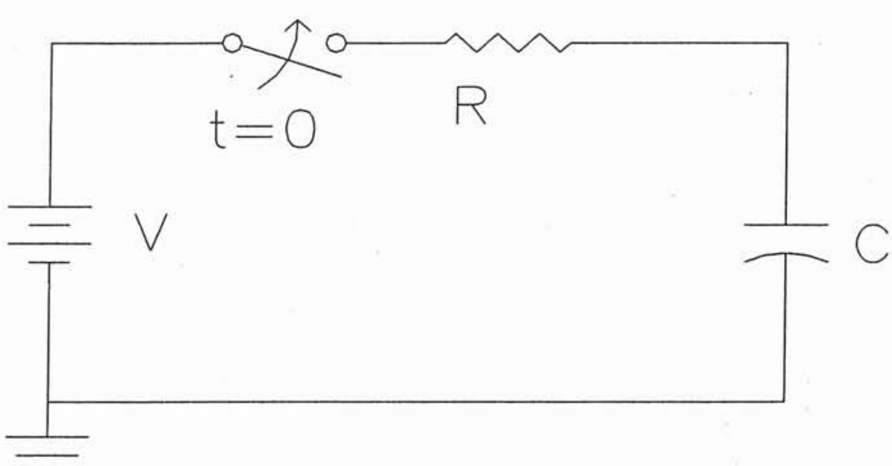
Laplace Ohm's Law	Impedance
$\frac{V_L(s)}{I(s)} = Ls$	$\frac{V_L}{I_L} = j \cdot \omega \cdot L$
$\frac{V_C(s)}{I(s)} = \frac{1}{C \cdot s}$	$\frac{V_C}{I_C} = \frac{1}{j \cdot \omega \cdot C}$
$\frac{V_R(s)}{I(s)} = R$	$\frac{V_R}{I_R} = R$

Solving Circuits and Systems Using the Laplace Transform Method

1.) Develop the differential equation model of the system.
2.) Transform equation using Laplace tables. Include voltage and current sources. (forces and torques)
3.) Solve resulting algebraic equations for variable(s) of interest. (usually a ratio of polynomials)
4.) For time response, take inverse Laplace transform. Results in form of exponential, sines and cosines.
5.) To find frequency response, replace s with jw and sweep frequency (note: take magnitude of complex quantity.

Step 4 gives total response to system: transient and steady-state, regardless of type of system excitation

Example: Find current through capacitor, $i_c(t)$



1.) Write integro-differential equation Using KVL

$$V = R \cdot i_c(t) + \frac{1}{C} \cdot \int_0^t i_c(t)\, dt$$

2.) Take Laplace transform of both sides of the equation Voltage source taken as a unit step with a magnitude of V volts.

$$\frac{V}{s} = R \cdot I_c(s) + \frac{1}{C \cdot s} \cdot I_c(s)$$

3.) Solve for $I_c(s)$

$$\frac{V}{s} = I_c(s) \cdot \left(R + \frac{1}{C \cdot s}\right)$$

Starts to look like phasor analysis

$$\frac{\frac{V}{s}}{\left(R + \frac{1}{C \cdot s}\right)} = I_c(s)$$

Simplify

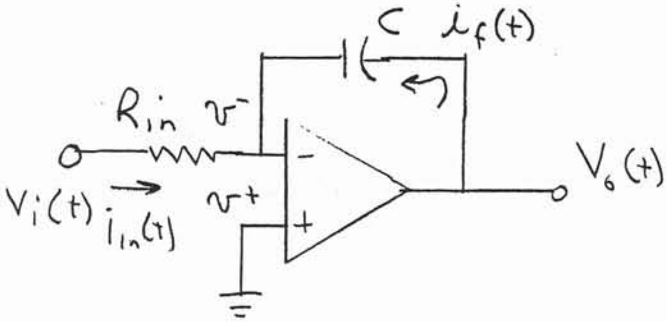
$$\frac{C \cdot s}{C \cdot s} \cdot \frac{\frac{V}{s}}{\left(R + \frac{1}{C \cdot s}\right)} = I_c(s)$$

$$C \cdot V \cdot \left(\frac{1}{R \cdot C \cdot s + 1}\right) = I_c(s)$$

Now make the terms inside the parenthesis look like expression in Laplace table

USING LAPLACE TO REPRESENT OP AMP CIRCUITS

INTEGRATOR



using OP AMP THEORY
no I Enters inverting node

$v^-(t) = v^+(t) = 0$

KCL at INVERTING NODE

$$i_{in}(t) + i_f(t) = 0 \Rightarrow i_{in}(t) = -i_f(t)$$

Currents in terms of node voltages

$$i_i(t) = \frac{V_i(t) - V^-(t)}{R_{in}} \quad i_f = C\frac{d}{dt}\left(V_o(t) - V^-(t)\right)$$

$V^-(t) = V^+(t) = 0$ so

$$i_i(t) = \frac{V_i(t)}{R_{in}} \quad i_f(t) = C\frac{d}{dt}V_o(t)$$

Substitute into KCL

$$\frac{V_i(t)}{R_{in}} = -C\frac{d}{dt}V_o(t)$$

integrate both sides
integration inverse of $\frac{d}{dt}$
to get $V_o(t)$

$$\frac{1}{R_{in}}\int V_i(t)\,dt = -C\,V_o(t)$$

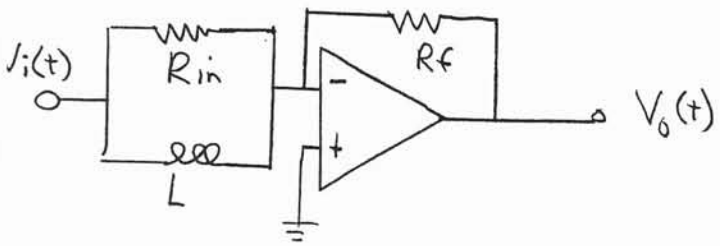
$$\frac{-1}{R_{in}C}\int V_i(t)\,dt = V_o(t) \quad \text{take Laplace}$$

$$-\frac{1}{R_{in}C}\,\frac{1}{s}\,V_i(s) = V_o(s)$$

($V_i(s)$ ↑ IN, $V_o(s)$ ↑ out)

CAN use GENERALIZED GAIN FORMULA of INVERTING AMP AND Laplace of Basic Relationships

Find the input/output relationship for the circuit below



Generalized inverting gain

$$\frac{V_o}{V_{in}} = \frac{-Z_f}{Z_{in}}$$

USE LAPLACE IMPEDANCE RELATIONSHIPS TO FIND GAIN

For L $V_L(t) = L\frac{d\,i_L(t)}{dt}$

$$\mathcal{L}\left[V_L(t)\right] = V_L(s) \quad \mathcal{L}\left[L\frac{d\,i_L(t)}{dt}\right] = LsI_L(s) \Rightarrow \frac{V_L(s)}{I_L(s)} = Ls$$

↑ LIKE X_L

so $Z_{in} = R_{in} \| Ls = \frac{R_{IN}\cdot(Ls)}{R_{IN} + Ls}$

$$Z_f = R_f \qquad \frac{V_o(s)}{V_{in}(s)} = \frac{-Z_f(s)}{Z_{in}(s)} = \frac{-R_f}{\frac{R_{IN}Ls}{R_{in}+Ls}} \quad \text{simplify}$$

$$\frac{V_o(s)}{V_{IN}(s)} = -R_f\left[\frac{R_{in}+Ls}{R_{in}(Ls)}\right] = \frac{-R_f}{R_{in}}\left[\frac{R_{IN}+Ls}{Ls}\right] = \left(\frac{R_{in}}{L}\left(\frac{1}{s}\right)+1\right)\left(\frac{-R_f}{R_{in}}\right)$$

↑ time constant

finally $V_o(s) = \frac{-R_f}{R_{in}}\left[\frac{R_{in}}{L}\left(\frac{1}{s}\right)+1\right]V_i(s)$

Output is sum of constant gain $\frac{-R_f}{R_{in}}V_i(s)$

and integrator action $\frac{-R_f}{R_{in}}\left(\frac{R_{in}}{L}\right)\left(\frac{1}{s}\right)V_i(s)$

Simplified form $-\frac{R_f}{Ls}V_i(s)$

↑ division by $s \rightarrow$ integration in time

Divide top and bottom by 1/RC

$$\frac{C\cdot V}{R\cdot C}\cdot\left(\frac{1}{s+\frac{1}{R\cdot C}}\right)=I_C(s)$$

Simplify and take inverse Laplace

$$\frac{V}{R}\cdot\mathcal{L}\left(\frac{1}{s+\frac{1}{R\cdot C}}\right)=\mathcal{L}\left(I_C(s)\right)$$

$$\mathcal{L}\left(e^{-a\cdot t}\right)=\frac{1}{s+a}$$

$$\frac{V}{R}\cdot e^{-\frac{t}{R\cdot C}}=i_C(t)$$

This matches the solution of the differential equation

For more complex Laplace expressions need to use Partial Fractions Expansion or Method of Residuals.

This breaks down the rational expression in s to a sum of simplier expressions in s with unknown coefficients

After finding coefficients, Inverse Laplace can be taken of the individual expressions to find the time response of the system.

Example 1: Find the inverse Laplace of the function below

$$H(s) = \frac{3\cdot s + 7}{s^2 - 2\cdot s - 3}$$

FACTOR DENOMINATOR AND EXPAND AS PARTIAL FRACTION

Find A

$$\frac{3s+7}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$$ Find A & B

mult. by each Factor evaluate result at root of factor

$$\left.\frac{(3s+7)(s-3)}{(s-3)(s+1)}\right|_{s=3} = \left.\frac{A(s-3)}{s-3} + \frac{B(s-3)}{s+1}\right|_{s=3}$$

$$\frac{3(3)+7}{3+1} = A + \frac{B(3-3)}{3+1} = \frac{9+7}{4} = A \Rightarrow \underline{A=4}$$

Find B

mult. By next factor $(s+1)$ evaluate remaining expression at root of factor $s=-1$

$$\left.\frac{(3s+7)(s+1)}{(s-3)(s+1)}\right|_{s=-1} = \left.\frac{A(s+1)}{s-3} + \frac{B(s+1)}{(s+1)}\right|_{s=-1}$$

$$\frac{(-1)3+7}{(-1-3)} = \frac{A(-1+1)}{-1-3} + B \rightarrow \frac{4}{-4} = B \quad \underline{B=-1}$$

$$\mathcal{L}^{-1}\left[\frac{3s+7}{(s-3)(s+1)}\right] = \mathcal{L}^{-1}\left[\frac{4}{s-3}\right] + \mathcal{L}^{-1}\left[\frac{-1}{s+1}\right]$$

$$\mathcal{L}^{-1}\left[\frac{3s+7}{(s-3)(s+1)}\right] = \underline{4e^{3t} - e^{-t}}$$ time function

Example 2: Find the inverse Laplace of the function

Find A

$$G(s) = \frac{2\cdot s^2 - 4}{(s+1)\cdot(s-2)\cdot(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3}$$

EVALUATE AT $(s+1)$

$$\left.\frac{(2s^2-4)(s+1)}{(s+1)(s-2)(s-3)}\right|_{s=-1} = \left.\frac{A(s+1)}{(s+1)} + \frac{B(s+1)}{(s-2)} + \frac{C(s+1)}{(s-3)}\right|_{s=-1}$$

$$\frac{2(-1)^2-4}{(-1-2)(-1-3)} = A + \frac{B(-1+1)}{-1-2} + \frac{C(-1+1)}{-1-3}$$ all terms but A drop out

$$\frac{-2}{(-3)(-4)} = A \Rightarrow \frac{-2}{12} = A \Rightarrow \boxed{A = -1/6}$$

Find B

$$\left.\frac{(2s^2-4)(s-2)}{(s+1)(s-2)(s-3)}\right|_{s=2} = \left.\frac{A(s-2)}{(s+1)} + \frac{B(s-2)}{(s-2)} + \frac{C(s-2)}{s-3}\right|_{s=2}$$

$$\frac{2(2)^2-4}{(2+1)(2-3)} = \frac{A(2-2)}{2+1} + B + \frac{C(2-2)}{2-3} \Rightarrow \frac{8-4}{3(-1)} = B \Rightarrow \boxed{\frac{-4}{3} = B}$$

Find C EVALUATE $(s-3)$

$$\left.\frac{(2s^2-4)(s-3)}{(s+1)(s-2)(s-3)}\right|_{s=3} = \left.\frac{A(s-3)}{s+1} + \frac{B(s-3)}{s-2} + \frac{C(s-3)}{(s-3)}\right|_{s=3}$$

$$\frac{(?)-4}{(?+1)(3-2)} = \frac{A(3-3)}{3+1} + \frac{B(3-3)}{3-2} + C \Rightarrow \frac{14}{4} = C \Rightarrow \boxed{7/2 = C}$$

$$\mathcal{L}^{-1}\left[\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right] = \mathcal{L}^{-1}\left[-1/6\left[\frac{1}{s+1}\right]\right] + \mathcal{L}^{-1}\left[\frac{-4}{3}\left[\frac{1}{s-2}\right]\right] + \mathcal{L}^{-1}\left[7/2\left[\frac{1}{s-3}\right]\right]$$

$$\mathcal{L}^{-1}\left[\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right] = -1/6\, e^{-t} - 4/3\, e^{2t} + 7/2\, e^{3t}$$ time response

Example 3: Complex roots in denominator. Find the inverse Laplace of the expression below.

$$G(s) = \frac{2\cdot s}{s^2 + 3\cdot s + 5}$$

Can't factor must use Quadratic formula or root solver in calculator

$a=1 \quad b=3 \quad c=5$

$$S = \frac{-3 \pm \sqrt{3^2 - 4(1)(5)}}{2(1)} = \frac{-3}{2} \pm j\frac{\sqrt{11}}{2}$$

ROOTS

$S = -1.5 \pm j1.6583$

Factored Form

$$\frac{2s}{(S+(1.5+j1.6583))(s+(1.5-j1.6583)}$$

expand as before

$$\frac{2s}{[S+(1.5+j1.6583)][S+(1.5-j1.6583)]} = \frac{A}{S+(1.5+j1.6583)} + \frac{B}{S+(1.5-j1.6583)}$$

Convert complex number to polar form to remove clutter

$1.5+j1.6583 = 2.236\angle 47.86^\circ \quad 1.5-j1.6583 = 2.236\angle -47.86^\circ$

$$\left.\frac{2s\,(S+2.236\angle 47.86^\circ)}{(S+2.236\angle 47.86^\circ)(S+2.236\angle -47.86^\circ)}\right|_{S=-2.236\angle 47.86^\circ} = \left.\frac{A(S+2.236\angle 47.86)}{S+2.236\angle 47.86^\circ} + \frac{B(S+2.236\angle 47.86^\circ)}{S+2.236\angle -47.86^\circ}\right|_{S=-2.236\angle 47.86}$$

$$\frac{2(-2.236\angle 47.86^\circ)}{-2.236\angle 47.86^\circ + 2.236\angle -47.86^\circ} = A + \frac{B(-2.236\angle 47.86^\circ + 2.236\angle 47.86^\circ)}{-2.236\angle 47.86^\circ + 2.236\angle -47.86^\circ}$$

$$\frac{-4.472\angle 47.86^\circ}{3.316\angle -90^\circ} = A$$

$$\boxed{1 - 0.9048j = A}$$

Example 3 (cont.)

Find B factor $(s+2.236\angle -47.86^\circ)$

$$\left.\frac{2s\,\cancel{(s+2.236\angle -47.86^\circ)}}{(s+2.236\angle 47.86^\circ)\cancel{(s+2.236\angle -47.86^\circ)}}\right|_{s=-2.236\angle -47.86^\circ} = \left.\frac{A(s+2.236\angle -47.86^\circ)}{s+2.236\angle 47.86^\circ}+\frac{B\cancel{(s+2.236\angle -47.86^\circ)}}{\cancel{(s+2.236\angle -47.86^\circ)}}\right|_{s=-2.236\angle -47.8}$$

$$\frac{2(-2.236\angle -47.86^\circ)}{-2.236\angle -47.86^\circ+2.236\angle 47.86^\circ} = \frac{A\overbrace{(-2.236\angle -47.86^\circ+2.236\angle -47.86^\circ)}^{0}}{-2.236\angle -47.86^\circ+2.236\angle 47.86^\circ}+B$$

$$\frac{-4.472\angle -47.86^\circ}{3.316\angle 90^\circ} = B \Rightarrow \boxed{B = 1+0.9048j} =$$

NOTE A & B are conjugate pair

$$\mathcal{L}^{-1}\left[\frac{2s}{s^2+3s+5}\right] = \mathcal{L}^{-1}\frac{(1-0.9048j)}{s+(1.5+j1.6583)}+\frac{(1+0.9048j)}{s+(1.5-j1.6583)}$$

$$= \mathcal{L}^{-1}\,(1-0.9048j)\left[\frac{1}{s+(1.5+j1.6583)}\right]+(1+0.9048j)\left[\frac{1}{s+(1.5-j1.6583)}\right]$$

$$= [1-0.9048j]\,e^{-(1.5+j1.6583)t}+(1+0.9048j)\,e^{-(1.5-j1.6583)t}$$

simplify

$$= [1-0.9048j]\,e^{(-1.5t)}e^{(-j1.6583t)}+(1+0.9048j)\,e^{(-1.5t)}e^{(+j1.6583t)}$$

$$= e^{-1.5t}\left[(1-0.9048j)e^{-j1.6583t}+(1+0.9048j)e^{j1.6583t}\right]$$

$$= e^{-1.5t}\left[(e^{-j1.6583t}+e^{j1.6583t})+0.9048j(e^{j1.6583t}-e^{-j1.6583t})\right]$$

From Euler's: $e^{j1.658t}+e^{-j1.658t} = 2\cos 1.658t$ $\quad e^{j1.658t}-e^{-j1.658t} = 2j\sin 1.658t$

$$= e^{-1.5t}\left[2\cos 1.6583t+(2j)(0.9048j)\sin 1.658t\right]$$

$$\boxed{= e^{-1.5t}\left[2\cos 1.6583t-1.8096\sin 1.658t\right]}$$