Lesson 19: Process Characteristics-
1\textsuperscript{st} Order Lag & Dead-Time Processes

ET 438a Automatic Control Systems
Technology

Learning Objectives

After this series of presentations you will be able to:

- Describe typical 1\textsuperscript{st} order lag and dead-time process models found in control systems.
- Write mathematical formulas for 1\textsuperscript{st} order and dead-time process models
- Compute the parameters of process models.
- Identify the Bode plots of typical process models.
- Identify the time response of typical process models.
First Order Lag Process

**Characteristics:**
1. Single storage element
2. Input produces an output related to amount of storage
3. Another name: self-regulating process

**Examples:**
- Series R-C circuit
- Series R-L circuit
- Self-regulating tank (valve on output)
- Tank heating

**Mathematical Descriptions**

**Time domain equation:**
\[ \tau \frac{dy}{dx} + y = G \cdot x \]

**Transfer function:**
\[ \frac{Y(s)}{X(s)} = \frac{G}{1 + \tau \cdot s} \]

**Equation Constants:**
- \( \tau \): Time constant of the system in seconds
- \( G \): Steady-state gain of the system
- \( t \): Time, in seconds
- \( y \): Output of process (units or %FS)
- \( x \): Input of process (units or %FS)

\( \tau \) determines time required for system to reach final value after step input

- \( 1\tau = 63.2\% \) Final value
- \( 5\tau = 99.3\% \) Final value

Where: \( G \) = steady-state gain of the system

\( t \) = time, in seconds

\( y \) = output of process (units or %FS)

\( x \) = input of process (units or %FS)

\( \tau \) = time constant of the system in seconds
First Order Lag Process - Self-Regulating Tank

Process Input

Process output: \( h \) = tank level, (% FS\(_{\text{out}}\))
FS\(_{\text{out}}\) = output range (m)

Process Parameters

- \( R_L \) = flow resistance
- \( C_L \) is the tank capacitance

Equation Constants

\[
\tau = R_L \cdot C_L \\
G = \frac{R_L}{\rho \cdot g} \left[ \frac{FS_{\text{in}}}{FS_{\text{out}}} \right]
\]

Where:
- \( \rho \) = liquid density (kg/m\(^3\))
- \( g \) = acceleration due to gravity 9.81 m/s\(^2\)

Example 19-1: Oil tank similar to previous figure has a diameter of 1.25 m and a height of 2.8 m. The outlet pipe is a smooth tube with a length of 5 m and diameter of 2.85 cm. Oil temperature 15 degrees C. The full scale flow rate is 24 L/min and full scale height is 2.8 m. Determine:

a) tank capacitance, \( C_L \)
b) pipe resistance, \( R_L \)
c) process time constant, \( \tau \)
d) process gain, \( G \)
e) time-domain equation
f) transfer function.
Example 19-1 Solution (1)

a) Find the tank capacitance

\[ C_L = \frac{A}{\rho g} \]
\[ A = \frac{\pi d^2}{4} = \frac{\pi (1.25\text{ m})^2}{4} = 1.23\text{ m}^2 \]

\[ C_L = \left[ \frac{1.23\text{ m}^2}{(880\text{ kg/m}^3)(9.81\text{ m/s}^2)} \right] = 1.425 \times 10^{-4}\text{ m}^3/\text{Pa} \]

Example 19-1 Solution (2)

b) Find the flow resistance. First compute the Reynolds number to determine flow type. Use maximum flow to determine it.

\[ R_e = \frac{\rho v d}{\mu} \quad \text{v = average velocity} \]
\[ \mu = \text{absolute viscosity} \]

\[ Q = 2.4\text{ L/min} \quad \text{Convert to m}^3/\text{s} \]

\[ Q = 2.4\text{ L/min} \left( 0.0167 \times 10^{-5}\text{ m}^3/\text{L} \right) = 4 \times 10^{-4}\text{ m}^3/\text{s} \]

\[ Q = VA \quad A_p = \text{Area of pipe} \]

Need pipe diameter for its area calculation

\[ d_p = 2.85\text{ cm} \]
\[ d_p = 2.85 \times 10^{-2}\text{ m} \]
Example 19-1 Solution (3)

\[ \frac{\pi (1.85 \times 10^{-2} \text{m})^2}{q} \]

\[ \frac{V = \frac{Q}{A} = \frac{4 \times 10^{-4} \text{m}^3/\text{s}}{6.379 \times 10^{-4} \text{m}^2}} = 0.627 \text{ m/s} \]

Now compute the Reynolds number

\[ \mu = 0.16 \text{ Pa-s} \]

\[ \frac{p = 880 \text{ kPa}} {1 \text{ m}^2} \]

\[ R = \frac{(880 \text{ kPa}) (0.627 \text{ m/s}) (2.85 \times 10^{-2} \text{ m})}{0.16 \text{ Pa-s}} \]

\[ R = \frac{16.725}{0.16} = 98.3 \text{ Laminar Flow } R < 2000 \]

Example 19-1 Solution (4)

For Laminar flow

\[ \frac{R_L = \frac{128 \mu L}{\pi d^4}} {\text{ Pa-s}/\text{m}^3} \]

\[ \frac{L = \text{ pipe length}} {\mu = 0.16 \text{ Pa-s}} \]

\[ R_L = \frac{128(0.16 \text{ Pa-s})(5 \text{ m})}{\pi (2.85 \times 10^{-2} \text{ m})^4} \]

\[ R_L = \frac{102.9}{(6.5775 \times 10^{-7})} = 4.94 \times 10^7 \text{ Pa-s}/\text{m}^3 \]

Now compute the tank time constant

\[ \gamma = \frac{R_L C_L}{1.94 \times 10^{-7} \text{ Pa-s}/\text{m}^3} \]

\[ \gamma (1.94 \times 10^{-7} \text{ Pa-s}/\text{m}^3) (1.926 \times 10^{-2} \text{ m}^2/\text{Pa}) \]

\[ \gamma = 7690 \text{ s} \]

\[ \gamma = 70.4 \text{ s} \times \frac{1 \text{ min}}{60 \text{ s}} = 1.17 \text{ min} \]

Tank level reduced to 63.2% of initial value after 117.3 minutes with \( q_{in} = 0 \). 99.2% empty after 5 \( \gamma \).
Example 19-1 Solution (5)

d) Compute process gain, \( G \)

\[
G = \frac{R_{C}}{R_{L}} \left[ \frac{F_{S_{in}}}{F_{S_{out}}} \right]
\]

\[F_{S_{in}} = 9 \times 10^{-9} \text{ m}^3/\text{s}\]
\[F_{S_{out}} = 2.8 \text{ m}\]
\[P = 880 \text{ KPa/m}^2\]
\[R_{L} = 2.97 \times 10^{7} \text{ Pa.s/m}^3\]

\[
G = \frac{4.97 \times 10^{-7} \text{ Pa.s/m}^3}{(880 \text{ KPa/m}^2)(9.81 \text{ m}^2)} \left[ \frac{4 \times 10^{-9} \text{ m}^3/\text{s}}{2.8 \text{ m}} \right]
\]

\[
G = 5.722.4 (1.429 \times 10^{-9}) = 0.818
\]

Example 19-1 Solution (6)
e) Find the differential equation for this process

General Equation

\[
x = q_{in} \quad \text{(in flow)}
\]
\[y = h \quad \text{(tank liquid)}
\]
\[\frac{d}{dt} + G \cdot q_{in} \quad \text{and} \quad h \quad \text{are in ft}
\]

\[
7.090 \frac{dh}{dt} + h = 0.818 q_{in}
\]

f) Find the transfer function for this process

General Equation

\[
\frac{H(s)}{Q_{in}(s)} = \frac{G}{1 + 85} = \frac{0.818}{1 + 70.90 s}
\]

Ans
Step Response and Bode Plots of The First-Order Lag Process

MatLAB Code

```matlab
% close all previous figures and clear all variables
close all;
clear all;
% input the integral time constant
Tl=input('enter the process time constant: ');
G=input('enter the gain of the process: ');
% construct and display the system
sys=tf(G,[Tl 1]);
sys
% plot the frequency response
bode(sys);
% construct a new figure and plot the time response
figure;
% define a range of time
t=(0:500:5*Tl);
% use it to generate a step response plot
step(sys,t);
```

Step Response of First-Order Lag Example 19-1

![Step Response Graph](image)
Frequency Response of First-Order Lag Processes

Similar to low pass filter plots.

Note the cutoff frequency, $f_c$, is when gain is -3 dB from passband $f = f_c$

$f_c = 1/\tau = 1.42 \times 10^{-4}$ rad/sec

First-Order Lag Process: Thermal Example

**Example 19-2:** Temperature of oil bath $\theta_L$ depends on the steam temperature $\theta_J$ and the thermal resistance and capacitance of the system. The equation below is the general model.

$$R_T \cdot C_T \left[ \frac{d\theta_L}{dt} \right] + \theta_L = \theta_J$$

Assume constant steam flow and temperature. The oil-filled tank is 1.2 m tall with a 1 m diameter. The inside film coefficient is 62 W/m²-K and the outside film coefficient is 310 W/m²-K. The tank is made of steel with a wall thickness of 1.2 cm. Find a) thermal resistance b) thermal capacitance, c) thermal time constant d) differential equation model, e) transfer function model.
Example 19-2 Solution (1)

a) Compute the thermal resistance

\[ k = \text{Thermal coefficient of steel} \]
\[ k_s = 45 \text{ W/m} \cdot \text{°K} \]
\[ h_v = 62 \text{ W/m}^2 \cdot \text{°K} \]
\[ h_o = 310 \text{ W/m}^2 \cdot \text{°K} \]
\[ x = \text{wall thickness} \]
\[ x = 1.2 \text{ cm} = 0.012 \text{ m} \]

Unit thermal resistance

\[ R = \frac{1}{h_v} + \frac{x}{k_s} + \frac{1}{h_o} \]
\[ R_u = \frac{1}{62 \text{ W/m}^2 \cdot \text{°K}} + \frac{0.012 \text{ m}}{45 \text{ W/m}^2 \cdot \text{°K}} + \frac{1}{310 \text{ W/m}^2 \cdot \text{°K}} \]
\[ R_u = 0.000163 + 2.467 \times 10^{-5} + 0.00323 \]
\[ R_u = 0.000325 \text{ m}^2 \cdot \text{°K}/\text{W} \]

Example 19-2 Solution (2)

Find the surface area of the tank. Assume a circular tank.

Define: \( A_1 = \text{area of tank bottom} \)
\( A_o = \text{area of tank sides} \)

\[ A_1 = \pi d_T \frac{z}{q} = \frac{\pi (1 \text{ m})^2}{q} \]
\[ A_1 = 0.2859 \text{ m}^2 \]

Area of sides is area of a cylinder

\[ A_2 = \pi (1 \text{ m})(1.2 \text{ m}) = 3.768 \text{ m}^2 \]
\[ A_T = A_1 + A_2 \]
\[ A_T = 0.2859 \text{ m}^2 + 3.768 \text{ m}^2 \]
\[ A_T = 4.0539 \text{ m}^2 \]
Example 19-2 Solution (3)

Compute total thermal resistance

\[ R_T = \frac{R_w}{A} = \frac{0.0192 \text{ } m^2 \text{ } K/\text{W}}{0.555 \text{ } \text{m}^2} = 0.0348 \text{ } K/\text{W} \]

b) Thermal capacitance

\[ C_T = mS_h \]

- \( m = \text{mass of oil} \)
- \( S_h = \text{specific heat of oil} \)
- \( S_l = 2180 \text{ } J/\text{kg} \cdot \text{K} \)

\[ \rho = \frac{\pi d^2}{4} h = \frac{\pi (12)^2}{4} (12.5) \text{ m}^3 \]

\[ V = 0.9425 \text{ m}^3 \]

\[ m = \rho V = 880 \text{ } \text{Kg}/\text{m}^3 (0.9425 \text{ m}^3) \]

\[ m = 829.9 \text{ } \text{Kg} \]

Example 19-2 Solution (4)

\[ C_T = \frac{(829.9 \text{ } \text{Kg}) (2180 \text{ } J/\text{K})}{1.808 \times 10^6 \text{ } J/\text{K}} \]

\[ C_T = 1.808 \times 10^6 \text{ } J/\text{K} \]

Use the values of \( R_T \) and \( C_T \) to find time constant

\[ \tau = \frac{R_T C_T}{(0.019 \text{ } 307 \text{ } K/\text{W})(1.808 \times 10^6 \text{ } J/\text{K})} \]

\[ \tau = 77.875 \text{ s} \]

Step change in input will take \( 5\tau \) to reach final value.

\[ 5\tau = 649 \text{ minutes (10.82 hours)} \]
Example 19-2 Solution (5)

d) Find the time function

\[ \gamma \frac{dy}{dt} + y = Gx \]

G=1 in this case so:

\[ x = \theta_L \]
\[ y = \theta_L \]
\[ \gamma \frac{d\theta_L}{dt} + \theta_L = \theta_d \]

e) Find the transfer function

\[ \frac{\theta_L(s)}{\theta_L(s)} = \frac{1}{1 + \gamma s} \]

Dead-Time Process

Characteristic: Energy or mass transported over a distance
Common in process industries (Chemicals Refining etc)

Time domain equation:

\[ f_o(t) = f_i(t - t_d) \]
\[ t_d = \frac{D}{v} \]

Transfer function:

\[ \frac{F_o(s)}{F_i(s)} = e^{-t_d s} \]

Where:

\[ f_o(t) = \text{output function} \]
\[ f_i(t) = \text{input function} \]
\[ v = \text{velocity of response travel} (\text{m/sec}) \]
\[ D = \text{distance from input to output} \]
\[ t_d = \text{dead-time lag (sec or minutes)} \]
\[ F_o(s) = \text{Laplace transform of output} \]
\[ F_i(s) = \text{Laplace transform of input} \]
Dead-Time Process

**Example: 19-3:** Determine the dead-time lag and the process transfer function if the salt-water solution travels at 0.85 m/sec and the distance to the bend is 15 m. Plot the time and frequency response of this system to a step-change in inlet concentration.

![Diagram showing salt-water solution flow with parameters and variables.](image)

**Example 19-3 Solution (1)**

1. **Define parameters**
   - $v = 0.85$ m/sec
   - $D = 15$ m

2. **Compute time delay**
   - $t_d = \frac{D}{v} = \frac{15 \text{ m}}{0.85 \text{ m/sec}} = 17.65$ sec

3. **Time function:**
   - $c_o(t) = c_i(t - t_d)$
   - $c_o(t) = c_i(t - 17.65)$

4. **Transfer function:**
   - $C_o(s) = C(s) e^{-17.65s}$

Now plot the time and frequency responses.
Dead-Time Process Time Plot

Dead-Time Process Time Response

\[ c(t) \]

Input Concentration
Output Concentration
Dead-Time Process Time Response
Time (Seconds)
Concentration Level (0-1)

\[ c_i(t) \]

\[ c_o(t) \]

t = 17.65 \text{ sec}

Dead-Time Frequency Bode Plot

Phase increases as frequency increases. Becomes very large for high frequencies.

Gain is constant over all frequencies \((0 \text{ dB } G=1)\)
End Lesson 19: Process Characteristics-1st Order and Dead-Time Processes
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