Lesson 20: Process Characteristics - 2nd Order Lag Process

ET 438a Automatic Control Systems
Technology

Learning Objectives

After this series of presentations you will be able to:

- Describe typical 2nd order lag models found in control systems.
- Write mathematical formulas for 2nd order lag process models.
- Compute the parameters of this process model.
- Identify the Bode plots of this process model.
- Identify the time response of this process model.
Second-Order Lag Processes

**Characteristics:**
- Two energy storage elements
- System response determined by three parameters: steady-state gain $G$, damping ratio $\zeta$, and resonant frequency $\omega_0$.

**Examples:**
- 2 capacitances,
- 1 mass and 1 spring
- 1 capacitance and 1 inductance

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General Second Order Lag Process Equations

**Time domain equation:**
$$A_2 \cdot \frac{d^2y}{dt^2} + A_1 \cdot \frac{dy}{dt} + y = G \cdot x$$

**Transfer function:**
$$\frac{Y(s)}{X(s)} = \frac{G}{1 + A_1 \cdot s + A_2 \cdot s^2}$$

**Parameters in terms of coefficients $A_1$ and $A_2$:**
$$\omega_0 = \sqrt{\frac{1}{A_2}} \quad \zeta = \frac{A_1}{2 \cdot \sqrt{A_2}} = \frac{A_1 \cdot \omega_0}{2}$$

**Coefficients in terms of Parameters $\zeta$ and $\omega_0$:**
$$A_2 = \frac{1}{\omega_0^2} \quad A_1 = \frac{2 \cdot \zeta}{\omega_0}$$
General Second Order Lag Process Equations

\[
\frac{Y(s)}{X(s)} = \frac{G}{1 + A_1 s + A_2 s^2}, \quad A_1 = \frac{1}{\omega_0}, \quad A_2 = \frac{2 \cdot \zeta}{\omega_0}
\]

Combine these two equations and simplify

\[
\frac{Y(s)}{X(s)} = \frac{\frac{\omega_0^2}{G} \cdot \left[ \frac{\omega_0^2}{s + \frac{2 \cdot \zeta}{\omega_0} s + \frac{1}{\omega_0} s^2} \right]}{1 + \left[ \frac{2 \cdot \zeta}{\omega_0} s + \frac{1}{\omega_0} s^2 \right]}
\]

Characteristic Equation: Roots determine system response

Second Order System Responses

Find roots to characteristic equation

\[
\omega_0^2 + 2 \cdot \zeta \cdot \omega_0 \cdot s + s^2 = 0
\]

\[
s_1 = -\zeta \cdot \omega_0 + j \cdot \omega_0 \cdot \sqrt{\zeta^2 - 1} = \omega_1 + j \cdot \omega_1
\]

\[
s_2 = -\zeta \cdot \omega_0 - j \cdot \omega_0 \cdot \sqrt{\zeta^2 - 1} = \omega_2 + j \cdot \omega_2
\]

Two poles at these locations

Plot these roots on complex plane

As poles near imaginary axis system become more oscillatory

If \( \zeta = 0 \), damping is zero and system will oscillate at \( \omega = \omega_0 \)
Second Order System Responses

Roots of quadratic formula can have three possible forms
1) real - distinct
2) real - identical
3) imaginary - conjugate pairs

Location of roots is controlled by the values of $\zeta$ and $\omega_0$. If natural frequency is constant then damping controls system response.
Second Order System Responses

Damping coefficient value and system responses

\[ \zeta = 1 \] - critically damped system. Reaches the final value the fastest without having any overshoot. Roots are equal and real.

Time response

Final value in approx. 1.4 seconds No Overshoot.

Second Order System Responses-Critically Damped

Bode plot of Critically damped system

Two poles at this point -3 dB from max. gain

Response similar to lag process
Second Order Response-Over Damped System

\( \zeta > 1 \) - over damped system. Reaches the final value slowly but with no overshoot. More damping, slower response to final value. Roots are real but not equal.

Compared to the critically damped case, the response time is slower. Approx. 6.5 sec to get to final value vs 1.4 sec.

Over Damped System Frequency Response

Bode plot of Over damped system

Pole at this point -3 dB from max. gain

Second Pole at higher frequency
Second Order Response-Under Damped System

\( \zeta < 1 \) - under damped system. Reaches the final value fast but with overshoot. Less damping more overshoot. Roots are conjugate pairs.

\[ z < 1 \]

Settling time is time required to reach 100% of final value. Approximately 4.5 sec.

Under Damped System Frequency Response

Bode plot of under damped system

- Resonant Peak at 5 rad/sec
- Natural oscillating frequency of system 90 degree phase shift at resonant frequency
Determining System Parameters From Characteristic Equation

Example 20-1: The block diagrams shown below represent three second order systems. Use the characteristic equations of each transfer function to determine the values of $\omega_0$ and $\zeta$ for each and determine if each system is over, under or critically damped.

a) $\frac{25}{s^2 + 10s + 25} = Y(s)$

b) $\frac{35}{s^2 + 30s + 25} = Y(s)$

c) $\frac{15}{s^2 + 2s + 25} = Y(s)$

Example 20-1 Solution (1)

a) Equate the general characteristic equation with that of the transfer function.

$$s^2 + 10s + 25 = s^2 + 2\cdot\zeta\cdot\omega_0 \cdot s + \omega_0^2$$

Equate Coefficients

$25 = \omega_0^2 \quad \sqrt{25} = \omega_0 = 5 \quad \text{Ans}$

$10 = 2\cdot\zeta\cdot\omega_0 \quad \text{so} \quad 10 = 2\cdot\zeta\cdot5$

$\zeta = 1 \quad \text{Ans} \quad \text{Critical damped}$

b) $\frac{Y(s)}{X(s)} = \frac{35}{s^2 + 30s + 25}$

$$s^2 + 30s + 25 = s^2 + 2\cdot\zeta\cdot\omega_0 \cdot s + \omega_0^2$$

$25 = \omega_0^2 \quad \sqrt{25} = \omega_0 = 5 \quad \text{Ans}$

$30 = 2\cdot\zeta\cdot\omega_0 \quad \text{so} \quad 30 = 2\cdot\zeta\cdot5$

$\zeta = 3 \quad \text{Ans} \quad \text{Over damped}$
Example 20-1 Solution (2)

c) \[ Y(s) = \frac{15}{X(s)} \frac{s^2 + 2s + 25}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \]

\[ s^2 + 2s + 25 = s^2 + 2\cdot\zeta\cdot\omega_0\cdot s + \omega_0^2 \]

\[ 25 = \omega_0^2 \quad \sqrt{25} = \omega_0 = 5 \quad \text{Ans} \]

\[ 2 = 2\cdot\zeta\cdot\omega_0 \quad \text{so} \quad 2 = 2\cdot\zeta\cdot5 \]

\[ \zeta = 0.2 \quad \text{Ans} \quad \text{Under damped} \]

Note: The numerator of the transfer functions does not affect the response of the system. It determines the maximum output.

As damping increases, one of the system poles becomes more dominant. The dominant pole controls system response.

Second Order Mechanical System

Example 20-2: The mechanical system shown below is at rest with an initial height of \( h(0) = 0 \). An external input force, \( f(t) \) disturbs the system. The system output, \( h(t) \) is the centerline position of the mass. The system parameters are:

- \( C_m = \) spring capacitance (Inverse of spring constant, \( K \)) = 0.001 m/N
- \( R_m = \) resistance due to viscous friction (\( B \)) = 20 N-s/m
- \( M = \) mass = 10 Kg

Find:
- a) coefficients of the second order system equation
- b) system transfer function
- c) resonant frequency
- d) damping ratio
- e) if system is over, under or critically damped
Example 20-2 Solution (1)

a) Sum the external forces. From Newton’s Law  \( F = Ma \)

Sum forces

\[
\sum f(t) = f(t) + f_a(t) + f_s(t)
\]

Where:

\[
f_s(t) = \frac{h(t)}{C_m}
\]

\[
f_a(t) = R_m \left( \frac{dh(t)}{dt} \right)
\]

\[
Ma = M \cdot \frac{d^2 h(t)}{dt^2}
\]

Example 20-2 Solution (2)

\[
f(t) - \frac{h(t)}{C_m} - R_m \left( \frac{dh(t)}{dt} \right) - M \cdot \frac{d^2 h(t)}{dt^2} = 0
\]

Add terms to both sides

\[
f(t) = M \frac{d^2 h(t)}{dt^2} + R_m \left( \frac{dh(t)}{dt} \right) + \frac{h(t)}{C_m}
\]

\[
C_m \cdot f(t) = C_m \cdot M \cdot \frac{d^2 h(t)}{dt^2} + C_m \cdot R_m \left( \frac{dh(t)}{dt} \right) + h(t)
\]

\[
G \cdot x(t) = A_x \cdot \frac{d^2 y(t)}{dt^2} + A_1 \left( \frac{dy(t)}{dt} \right) + y(t)
\]

General form of 2nd order system

\[
A_x = C_m \cdot M = (0.001 \text{ m/N})(10 \text{ Kg}) = 0.01 \text{ sec}^2
\]

\[
A_1 = C_m \cdot R_m = (0.001 \text{ m/N})(20 \text{ N/ sec/m}) = 0.02 \text{ sec}
\]

\[
G = C_m = 0.001 \text{ m/N}
\]
Example 20-2 Solution (2)

b) Find transfer function

\[ 0.01 \cdot \frac{d^2 h(t)}{dt^2} + 0.02 \left( \frac{dh(t)}{dt} \right) + h(t) = 0.001 \cdot f(t) \]

Take Laplace transform of above equation and factor out common term

\[ 0.01 \cdot s^2 \cdot H(s) + 0.02 \cdot s \cdot H(s) + H(s) = 0.001 \cdot F(s) \]

\[ H(s) = \frac{0.001 \cdot F(s)}{(0.01 \cdot s^2 + 0.02 \cdot s + 1)} \]

\[ H(s) = \frac{0.001}{(0.01 \cdot s^2 + 0.02 \cdot s + 1)} \]

\[ F(s) = \frac{0.001}{(0.01 \cdot s^2 + 0.02 \cdot s + 1)} \]

Ans

Example 20-2 Solution (3)

c) Find the resonant frequency \( \omega_0 \)

\[ \omega_0 = \frac{1}{\sqrt{A_2}} = \frac{1}{\sqrt{0.01 \text{sec}^2}} = 10 \text{ rad/sec} \]

Ans

d) Find the damping ratio \( \zeta \)

\[ \zeta = \frac{A_1}{2 \cdot \sqrt{A_2}} = \frac{0.02 \text{ sec}}{2 \cdot \sqrt{0.01 \text{sec}^2}} = 0.1 \]

Ans

e) The damping ratio is \( \zeta < 1 \), so the system is under damped. Applying external force causes the mass center to oscillate before coming to rest at a new position.
End Lesson 20: Process Characteristics-2nd Order Process

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