Lesson 6: Mathematical Models of Fluid Flow Components

ET 438a
Automatic Control Systems Technology

Learning Objectives

After this presentation you will be able to:

- Define the characteristics of a fluid flow system
- Identify if a fluid flow is Laminar or Turbulent based on fluid and system parameters.
- Write mathematical models for fluid characteristics.
- Develop an analogy between electrical characteristic and fluid system characteristics.
- Solve for steady-state fluid flow using given mathematical modeling equations.
Liquid Flow Characteristics

Flow
Volume of liquid/unit time (e.g. gallons/minute)
Current (Coulombs/s)

Pressure
force/unit area (e.g. lb/in²)
Potential Difference

Flow Resistance
Pressure drop/unit flow (e.g. lb/in²/gal)
Electrical Resistance

Note: Flow resistance is either linear or non-linear. It depends on type of fluid flow, which is based on the fluid and piping parameters.

Fluid Flow Classifications

Types of liquid flow

Laminar Flow - low velocity flows. Stream lines are parallel. Liquid flows in layers. Linear flow resistance.
**Fluid Flow Classifications**

Types of liquid flow

**Turbulent** - relatively high velocity flow. Liquid swirls and spins as it flows. Non-linear flow resistance.

Flow type determined by the Reynold's Number

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**Determining Flow Types**

Reynold's Number

$$R = \frac{\rho \cdot v \cdot d}{\mu}$$

Where
- $\rho$ = density of the fluid (kg/m$^3$)
- $v$ = average velocity of the fluid (m/s)
- $d$ = diameter of pipe (m)
- $\mu$ = absolute viscosity of fluid (Pa-s)

Note: Reynold's number is dimensionless

- **Laminar flow**: $R<2000$
- **Turbulent flow**: $R>4000$
- **Transition flow**: $2000 < R < 4000$
Determining Flow Types

Computing average velocity

Laminar Flow

Velocity profile changes across the cross section of pipes and ducts

\[ v = \frac{Q}{A} \]

Since pipe diameter is usually given

\[ v = \frac{4 \cdot Q}{\pi \cdot d^2} \]

Where:  
- \( A \) = area of pipe (m\(^2\))  
- \( Q \) = flow (m\(^3\)/s)  
- \( d \) = pipe diameter (m)

Laminar Flow Equations

Laminar Flow Equations for Round Pipes

\[ P = R_L \cdot Q \]  \hspace{0.5cm} (Pa)

\[ R_L = \frac{128 \cdot \mu \cdot l}{\pi \cdot d^4} \]

Where:  
- \( P \) = pressure drop (Pascals)  
- \( R_L \) = laminar flow resistance  
- \( Q \) = flow (m\(^3\)/s)  
- \( l \) = length of pipe (m)  
- \( \mu \) = absolute viscosity (Pa-s)  
- \( d \) = pipe diameter (m)
Turbulent Flow Equations

Turbulent Flow Equations for Round Pipes

\[ P = K_t \cdot Q^2 \]  \hspace{1cm} (Pa)

\[ K_t = \frac{8 \cdot \rho \cdot f \cdot l}{\pi^2 \cdot d^5} \]  \hspace{1cm} (Pa \cdot s/m^3)

\[ R_t = 2 \cdot K_t \cdot Q \]

Where:
- \( f \) = friction factor  \hspace{0.5cm} (see table 3.3 p. 81 text)
- \( l \) = length (m)
- \( d \) = pipe diameter (m)
- \( \rho \) = density of liquid (kg/m^3)
- \( R_t \) = turbulent flow resistance (Pa \cdot s/m^3)
- \( p \) = pressure (Pa)
- \( Q \) = flow (m^3/s)

Laminar Flow Example

**Example 6-1:** Oil at a temperature of 15 C flows in a horizontal, 1 cm diameter tube with a flow rate of 9.42 L/min. Tube length is 10 m. Find: Reynolds number, flow resistance, pressure drop in tube.

Convert all units into SI units

\[ d = 1 \text{ cm} = 0.01 \text{ m} \]
\[ Q = 9.42 \text{ L/min} \left( \frac{1 \text{ L}}{60 \text{ s}} \right) \left( \frac{1 \text{ m}^3}{1000 \text{ L}} \right) = 1.57 \times 10^{-4} \text{ m}^3/\text{s} \]

Average velocity

\[ \bar{v} = \frac{4Q}{\pi d^2} = \frac{4 (1.57 \times 10^{-4} \text{ m}^3/\text{s})}{\pi (0.01 \text{ m})^2} \approx 2.0 \text{ m/s} \]

From Appendix A in textbook

\[ \rho = 880 \text{ kg/m}^3 \hspace{0.5cm} \mu = 0.160 \text{ Pa-s} \]

Compute Reynolds number

\[ Re = \frac{\rho \bar{v} d}{\mu} = \frac{(880 \text{ kg/m}^3)(2.0 \text{ m/s})(0.01 \text{ m})}{(0.160 \text{ Pa-s})} = 1.09 \times 10^5 \]

R < 2000 so flow in Laminar. Linear relationship between flow and pressure drop.
Laminar Flow Solution (2)

Compute Laminar flow resistance

\[
R_L = \frac{128 \mu_1}{
\pi \left(\frac{d}{2}\right)^4}
\]

\[
R_L = \frac{128 \times (0.010 \text{Pa-s}) \times (10 \text{m})}{
\pi \times (0.010 \text{m})^4}
\]

\[
R_L = 6.519 \times 10^9 \text{ Pa-s/m}^2
\]

Compute pressure drop

\[
\Delta p = R_L Q
\]

\[
\Delta p = (6.519 \times 10^9 \text{ Pa-s/m}^2) \times (6 \times 10^{-3} \text{ m}^3/\text{s})
\]

\[
\Delta p = 1.0285 \times 10^4 \text{ Pa}
\]

Convert to psi

\[
\Delta p = \frac{1.0285 \times 10^4 \text{ Pa}}{1 \text{ psi}} = 148.9 \text{ psi}
\]

Turbulent Flow Example

**Example 6-2:** Water at 15°C flows through a commercial steel pipe with a diameter of 0.4 inch with a flow rate of 6 gal/min. The line is 50 ft long. Find: Reynolds’s number, flow resistance, pressure drop in pipe.

From Appendix A: Density \( \rho = 1000 \text{ kg/m}^3 \) Viscosity \( \mu = 0.001 \text{ Pa-s} \)

Convert all English units to SI units

- \( d = (0.4 \text{ in}) \times (0.0254 \text{ m/in}) = 0.01016 \text{ m} \)
- \( Q = (6 \text{ gal/min}) \times (6.3088 \times 10^{-5} \text{ m}^3/\text{L}) = 3.7853 \times 10^{-3} \text{ m}^3/\text{s} \)
- \( l = (50 \text{ ft}) \times (0.3048 \text{ m/ft}) = 15.24 \text{ m} \)

Find the average velocity

\[
V = \frac{Q}{\pi d^2} = \frac{4 \times (3.7853 \times 10^{-3} \text{ m}^3/\text{s})}{\pi \left(\frac{0.01016 \text{ m}}{2}\right)^2} = 4.669 \text{ m/s}
\]
Turbulent Flow Solution (2)

Compute the Reynolds number
\[ R = \frac{\rho V d}{\mu} = \frac{(1000 \text{ kg/m}^3)(4.69 \text{ m/s})(0.0166 \text{ m})}{0.001 \text{ Pa-s}} \]
\[ R = 47,448 \quad R > 2000 \]

R > 2000 so the flow is turbulent. Use the turbulent equations.

Turbulent equations require friction factors from Table 3-3 on page 81 of textbook.

\[ P = K_t Q^2 \]
Find \( f \) values from Table 3-3. Get values that bracket the Computed value.

For commercial steel pipe: diameter 1-2 cm

\[ R \text{ between } R_a > 10,000 \quad R_b < 100,000 \]
\[ f \text{ between } f_a = 0.035 \quad f_b = 0.028 \]

Turbulent Flow Solution (3)

Friction factor, \( f \), must be between \( f_a \) and \( f_b \)

\[ f = \frac{0.035 + (0.028 - 0.035)}{2} \quad \frac{0.035 + (0.028 - 0.035)}{2} = 0.035 \]
\[ f = 0.035 + 0.009 \left( \frac{3.724}{9} \right) \]
\[ f = 0.032 \]

Now find \( K_t \)

\[ K_t = \frac{8 \pi f Q L}{13.85} \]
\[ K_t = 3.44 \times 10^{-12} \text{ Pa-s/m}^3 \]

Find \( R_t \)

\[ R_t = 2 K_t Q \]
\[ R_t = 2 \left( 3.44 \times 10^{-12} \text{ Pa-s/m}^3 \right) (3.724 \times 10^{-9} \text{ m}^3) \]
\[ R_t = 2.719 \times 10^9 \text{ Pa-s/m}^3 \]
Turbulent Flow Solution (4)

Convert $R_e$ to English units

$$R_e = (2.77 \times 10^9 \text{ Pa}^{-1} \text{ m}^2) (9.148 \times 10^6) = 25.35 \text{ psi}^2/\text{gpm}$$

Compute pressure drop

$$P = R_e Q^2$$

$$P = (3.4619 \times 10^{-7} \text{ Pa}^{-1} \text{ m}^2) (3.7854 \times 10^3 \text{ m}^3/\text{s})$$

$$P = 5.24 \times 10^5 \text{ Pa}$$

Convert $P$ to English units

$$P = (5.24 \times 10^5 \text{ Pa}) (145 \times 10^2 \text{ psi/Pa})$$

$$P = 76.02 \text{ psi}$$

Liquid Flow Capacitance

**Liquid Flow Capacitance** - increase in volume of liquid required to produce unit increase in pressure

$$C_L = \frac{\Delta V}{\Delta P}$$

Where: $C_L =$ capacitance ($\text{m}^3/\text{Pa}$)

$\Delta V =$ volume change ($\text{m}^3$)

$\Delta p =$ pressure change (Pa)

Derive relationship for $C_L$.

Pressure relationship

$$\Delta P = \rho \cdot g \cdot \Delta H$$

Where: $\rho =$ density of fluid

$g =$ acceleration due to gravity

$H =$ height of liquid in tank
Liquid Flow Capacitance Derivation

Find liquid flow capacitance in terms of tank parameters.

\[ \Delta P = \rho \cdot g \cdot \Delta H \]
\[ \Delta H = \frac{\Delta V}{A} \]
\[ \Delta P = \rho \cdot g \left( \frac{\Delta V}{A} \right) \]
\[ C_L = \frac{\Delta V}{\Delta P} = \frac{\Delta V}{\rho \cdot g \left( \frac{\Delta V}{A} \right)} = \frac{A}{\rho \cdot g} \quad (m^3 / Pa) \]

\[ C_L = \frac{A}{\rho \cdot g} \quad (m^3 / Pa) \]

Fluid Capacitance Examples

**Example 6-3:** A tank has a diameter of 1.83 meters and a height of 10 ft. Determine the capacitance of the tank when it holds: a.) water b.) oil c.) kerosene d.) gasoline

\[ A = \frac{\pi \cdot d^2}{4} = \frac{\pi \cdot (1.83 \text{ m})^2}{4} \]
\[ A = 2.63 \text{ m}^2 \]
\[ \rho_w = 1000 \text{ kg/m}^3 \]
\[ C_{L,w} = \frac{2.63 \text{ m}^2}{1000 \text{ kg/m}^3 \cdot 9.81 \text{ m/s}^2} \]
\[ C_{L,w} = 2.68 \times 10^{-4} \text{ m}^3/\text{Pa} \]

\[ \rho_o = 880 \text{ kg/m}^3 \]
\[ C_{L,o} = \frac{2.63 \text{ m}^2}{(880 \text{ kg/m}^3 \cdot 9.81 \text{ m/s}^2)} \]
\[ C_{L,o} = 3.05 \times 10^{-3} \text{ m}^3/\text{Pa} \]
Fluid Capacitance Examples

Parts c. and d.

\[
C_L = \frac{2.43 \, m^2}{(800 \, kg/m^3) \cdot (9.8 \, m/s^2)} 
\]

\[
C_L = 3.35 \times 10^{-3} \, m^3/Pa
\]

\[
C_L = \frac{2.43 \, m^2}{(740 \, kg/m^3) \cdot (9.8 \, m/s^2)} = C_L
\]

\[
C_L = 3.62 \times 10^{-3} \, m^3/Pa = C_L
\]

Note: As density of fluid decreases, the volume of liquid required to produce a unit increase in pressure increases.

Fluid Inertance

Amount of pressure drop required to increase flow rate by one unit/second. Analogy-electrical inductance.

\[
I_L = \frac{p}{\Delta Q/\Delta t}
\]

Where \(I_L\) = inertance \((Pa/(m^3/s^2))\)
\(p\) = pressure drop \((Pa)\)
\(\Delta Q/\Delta t\) = change in flow

Inertance defined using physical parameters of pipe.

\[
I_L = \frac{\rho \cdot l}{A}
\]

Where:
\(A\) = area of pipe \((m^2)\)
\(\rho\) = density of liquid \((Kg/m^3)\)
\(l\) = length of pipe \((m)\)
Dead-Time Delay

**Dead-time Delay of Liquid** - time required to transport liquid from one point to another in piping system or ducts.

\[ t_d = \frac{D}{v} \]

- \( v \) = average velocity of fluid (m/s)
- \( D \) = distance traveled (m)

**Example 6-4:** Determine the inertance of water in a pipe with a diameter of 2.1 cm and a length of 65 meters.

- \( d = 2.1 \text{ cm} = 0.021 \text{ m} \)
- \( \rho = 1000 \text{ kg/m}^3 \)
- \( l = 65 \text{ m} \)

\[ I_t = \frac{\pi d^4}{16} \]

\[ A = \frac{\pi d^2}{4} \]

\[ I_t = \frac{\pi (0.021)^4}{16} = \frac{3.46 \times 10^{-9}}{16} \text{ m}^2 \]

\[ A = \frac{\pi (0.021)^2}{4} = \frac{3.46 \times 10^{-8}}{4} \text{ m}^2 \]

\[ I_t = 1.88 \times 10^{-8} \text{ Pa m}^3/\text{s}^2 \]

END LESSON 6:
MATHEMATICAL MODELS
OF FLUID FLOW
COMPONENTS